Mathematical Analysis

VOLUME I

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* "Starred" sections may be omitted by beginners.

Preface

This text is an outgrowth of lectures given at the University of Windsor, Canada. One of our main objectives is *updating* the undergraduate analysis as a rigorous postcalculus course. While such excellent books as Dieudonné's *Foundations of Modern Analysis* are addressed mainly to graduate students, we try to simplify the modern Bourbaki approach to make it accessible to sufficiently advanced undergraduates. (See, for example, §4 of Chapter 5.)

On the other hand, we endeavor not to lose contact with classical texts, still widely in use. Thus, unlike Dieudonné, we retain the classical notion of a derivative as a *number* (or vector), not a linear transformation. Linear maps are reserved for later (Volume II) to give a modern version of *differentials*. Nor do we downgrade the classical mean-value theorems (see Chapter 5, §2) or Riemann–Stieltjes integration, but we treat the latter rigorously in Volume II, inside Lebesgue theory. First, however, we present the modern Bourbaki theory of *antidifferentiation* (Chapter 5, §5 ff.), adapted to an undergraduate course.

Metric spaces (Chapter 3, §11 ff.) are introduced cautiously, after the n-space E^n, with simple diagrams in E^2 (rather than E^3), and many "advanced calculus"-type exercises, along with only a few topological ideas. With some adjustments, the instructor may even limit all to E^n or E^2 (but not just to the real line, E^1), postponing metric theory to Volume II. *We do not hesitate to deviate from tradition if this simplifies cumbersome formulations*, unpalatable to undergraduates. Thus we found useful some *consistent*, *though not very usual*, *conventions* (see Chapter 5, §1 and the end of Chapter 4, §4), and an *early use of quantifiers* (Chapter 1, §1–3), even in formulating theorems. Contrary to some existing prejudices, quantifiers are easily grasped by students after some exercise, and help clarify all essentials.

Several years' class testing led us to the following conclusions:

(1) Volume I can be (and *was*) taught even to sophomores, though they only gradually learn to *read* and *state* rigorous arguments. A sophomore often does not even know how to *start* a proof. The main stumbling block remains the ε, δ-procedure. As a remedy, we provide most exercises with explicit hints, sometimes with almost complete solutions, leaving only tiny "whys" to be answered.

(2) Motivations are good if they are brief and avoid terms not yet known. Diagrams are good if they are *simple* and appeal to intuition.

(3) Flexibility is a must. One must adapt the course to the level of the class. "Starred" sections are best deferred. (Continuity is not affected.)

(4) "Colloquial" language fails here. We try to keep the exposition rigorous and *increasingly concise*, but readable.

(5) It is advisable to make the students *preread* each topic and prepare questions in advance, to be answered *in the context* of the next lecture.

(6) Some topological ideas (such as compactness in terms of open coverings) are hard on the students. Trial and error led us to emphasize the sequential approach instead (Chapter 4, §6). "Coverings" are treated in Chapter 4, §7 ("starred").

(7) To students unfamiliar with elements of set theory we recommend our *Basic Concepts of Mathematics* for supplementary reading. (At Windsor, this text was used for a preparatory first-year one-semester course.) The first two chapters and the first ten sections of Chapter 3 of the present text are actually summaries of the corresponding topics of the author's *Basic Concepts of Mathematics*, to which we also relegate such topics as the construction of the real number system, etc.

For many valuable suggestions and corrections we are indebted to H. Atkinson, F. Lemire, and T. Traynor. Thanks!

Publisher's Notes

Chapters 1 and 2 and §§1–10 of Chapter 3 in the present work are summaries and extracts from the author's *Basic Concepts of Mathematics*, also published by the Trillia Group. These sections are numbered according to their appearance in the first book.

Several annotations are used throughout this book:

* This symbol marks material that can be omitted at first reading.

⇒ This symbol marks exercises that are of particular importance.

About the Author

Elias Zakon was born in Russia under the czar in 1908, and he was swept along in the turbulence of the great events of twentieth-century Europe.

Zakon studied mathematics and law in Germany and Poland, and later he joined his father's law practice in Poland. Fleeing the approach of the German Army in 1941, he took his family to Barnaul, Siberia, where, with the rest of the populace, they endured five years of hardship. The Leningrad Institute of Technology was also evacuated to Barnaul upon the siege of Leningrad, and there he met the mathematician I. P. Natanson; with Natanson's encouragement, Zakon again took up his studies and research in mathematics.

Zakon and his family spent the years from 1946 to 1949 in a refugee camp in Salzburg, Austria, where he taught himself Hebrew, one of the six or seven languages in which he became fluent. In 1949, he took his family to the newly created state of Israel and he taught at the Technion in Haifa until 1956. In Israel he published his first research papers in logic and analysis.

Throughout his life, Zakon maintained a love of music, art, politics, history, law, and especially chess; it was in Israel that he achieved the rank of chess master.

In 1956, Zakon moved to Canada. As a research fellow at the University of Toronto, he worked with Abraham Robinson. In 1957, he joined the mathematics faculty at the University of Windsor, where the first degrees in the newly established Honours program in Mathematics were awarded in 1960. While at Windsor, he continued publishing his research results in logic and analysis. In this post-McCarthy era, he often had as his house-guest the prolific and eccentric mathematician Paul Erdős, who was then banned from the United States for his political views. Erdős would speak at the University of Windsor, where mathematicians from the University of Michigan and other American universities would gather to hear him and to discuss mathematics.

While at Windsor, Zakon developed three volumes on mathematical analysis, which were bound and distributed to students. His goal was to introduce rigorous material as early as possible; later courses could then rely on this material. We are publishing here the latest complete version of the second of these volumes, which was used in a two-semester class required of all second-year Honours Mathematics students at Windsor.

Chapter 1

Set Theory

§§1–3. Sets and Operations on Sets. Quantifiers

A *set* is a collection of objects of any specified kind. Sets are usually denoted by capitals. The objects belonging to a set are called its *elements* or *members*. We write $x \in A$ if x is a member of A, and $x \notin A$ if it is not.

$A = \{a, b, c, \ldots\}$ means that A consists of the elements $a, b, c, \ldots$. In particular, $A = \{a, b\}$ consists of a and b; $A = \{p\}$ consists of p alone. The *empty* or *void* set, $\emptyset$, has *no* elements. Equality $(=)$ means *logical identity*.

If all members of A are also in B, we call A a *subset* of B (and B a *superset* of A), and write $A \subseteq B$ or $B \supseteq A$. It is an axiom that *the sets A and B are equal* $(A = B)$ *if they have the same members*, i.e.,

$$A \subseteq B \textit{ and } B \subseteq A.$$

If, however, $A \subseteq B$ but $B \nsubseteq A$ (i.e., B has some elements *not* in A), we call A a *proper* subset of B and write $A \subset B$ or $B \supset A$. "$\subseteq$" is called the *inclusion relation*.

Set equality is not affected by the *order* in which elements appear. Thus $\{a, b\} = \{b, a\}$. Not so for *ordered pairs* (a, b).[1] For such pairs,

$$(a, b) = (x, y) \quad \text{iff}^2 \quad a = x \text{ and } b = y,$$

but not if $a = y$ and $b = x$. Similarly, for *ordered n-tuples*,

$$(a_1, a_2, \ldots, a_n) = (x_1, x_2, \ldots, x_n) \quad \text{iff} \quad a_k = x_k,\ k = 1, 2, \ldots, n.$$

We write $\{x \mid P(x)\}$ for "the set of all x satisfying the condition $P(x)$." Similarly, $\{(x, y) \mid P(x, y)\}$ is the set of all *ordered pairs* for which $P(x, y)$ holds; $\{x \in A \mid P(x)\}$ is the set of those x *in* A for which $P(x)$ is true.

[1] See Problem 6 for a definition.

[2] Short for *if and only if*; also written $\Longleftrightarrow$.

For any sets A and B, we define their *union* $A \cup B$, *intersection* $A \cap B$, *difference* $A - B$, and *Cartesian product* (or *cross product*) $A \times B$, as follows:

$A \cup B$ is the set of all members of A and B taken *together*:

$$\{x \mid x \in A \text{ or } x \in B\}.^3$$

$A \cap B$ is the set of all *common* elements of A and B:

$$\{x \in A \mid x \in B\}.$$

$A - B$ consists of those $x \in A$ that are *not* in B:

$$\{x \in A \mid x \notin B\}.$$

$A \times B$ is the set of all *ordered pairs* (x, y), with $x \in A$ and $y \in B$:

$$\{(x, y) \mid x \in A, y \in B\}.$$

Similarly, $A_1 \times A_2 \times \cdots \times A_n$ is the set of all *ordered n-tuples* $(x_1, \ldots, x_n)$ such that $x_k \in A_k$, $k = 1, 2, \ldots, n$. We write A^n for $A \times A \times \cdots \times A$ (n factors).

A and B are said to be *disjoint* iff $A \cap B = \emptyset$ (no common elements). Otherwise, we say that A *meets* B ($A \cap B \neq \emptyset$). Usually all sets involved are subsets of a "*master set*" S, called the *space*. Then we write $-X$ for $S - X$, and call $-X$ the *complement* of X (in S). Various other notations are likewise in use.

Examples.

Let $A = \{1, 2, 3\}$, $B = \{2, 4\}$. Then

$$A \cup B = \{1, 2, 3, 4\}, \quad A \cap B = \{2\}, \quad A - B = \{1, 3\},$$
$$A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}.$$

If N is the set of all *naturals* (positive integers), we could also write

$$A = \{x \in N \mid x < 4\}.$$

Theorem 1.

(a) $A \cup A = A$; $A \cap A = A$;

(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$;

(c) $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap C = A \cap (B \cap C)$;

(d) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$;

(e) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

[3] The word "or" is used in the *inclusive* sense: "P or Q" means "P or Q or *both*."

The proof of (d) is sketched in Problem 1. The rest is left to the reader.

Because of (c), we may omit brackets in $A \cup B \cup C$ and $A \cap B \cap C$; similarly for four or more sets. More generally, we may consider whole *families* of sets, i.e., collections of many (possibly infinitely many) sets. If $\mathcal{M}$ is such a family, we define its *union*, $\bigcup \mathcal{M}$, to be the set of all elements x, each belonging to *at least one* set of the family. The intersection of $\mathcal{M}$, denoted $\bigcap \mathcal{M}$, consists of those x that belong to all sets of the family *simultaneously*. Instead, we also write

$$\bigcup \{X \mid X \in \mathcal{M}\} \text{ and } \bigcap \{X \mid X \in \mathcal{M}\}, \text{ respectively.}$$

Often we can *number* the sets of a given family:

$$A_1, A_2, \ldots, A_n, \ldots.$$

More generally, we may denote all sets of a family $\mathcal{M}$ by some letter (say, X) with indices i attached to it (the indices may, but *need not*, be numbers). The family $\mathcal{M}$ then is denoted by $\{X_i\}$ or $\{X_i \mid i \in I\}$, where i is a variable index ranging over a suitable set I of indices ("index notation"). In this case, the union and intersection of $\mathcal{M}$ are denoted by such symbols as

$$\bigcup \{X_i \mid i \in I\} = \bigcup_i X_i = \bigcup X_i = \bigcup_{i \in I} X_i;$$
$$\bigcap \{X_i \mid i \in I\} = \bigcap_i X_i = \bigcap X_i = \bigcap_{i \in I} X_i.$$

If the indices are *integers*, we may write

$$\bigcup_{n=1}^{m} X_n, \; \bigcup_{n=1}^{\infty} X_n, \; \bigcap_{n=k}^{m} X_n, \text{ etc.}$$

Theorem 2 (De Morgan's duality laws)**.** *For any sets S and A_i ($i \in I$), the following are true*:

$$\text{(i) } S - \bigcup_i A_i = \bigcap_i (S - A_i); \quad \text{(ii) } S - \bigcap_i A_i = \bigcup_i (S - A_i).$$

(If S is the entire space, we may write $-A_i$ for $S - A_i$, $-\bigcup A_i$ for $S - \bigcup A_i$, etc.)

Before proving these laws, we introduce some useful notation.

Logical Quantifiers. From logic we borrow the following abbreviations.

"$(\forall\, x \in A)$ ..." means "For each member x of A, it is true that ...."

"$(\exists\, x \in A)$ ..." means "There is at least one x in A such that ...."

"$(\exists!\, x \in A)$ ..." means "There is a *unique* x in A such that ...."

The symbols "$(\forall\, x \in A)$" and "$(\exists\, x \in A)$" are called the *universal* and *existential quantifiers*, respectively. If confusion is ruled out, we simply write "$(\forall\, x)$," "$(\exists\, x)$," and "$(\exists\,!\, x)$" instead. For example, if we agree that m, n denote *naturals*, then

$$\text{“}(\forall\, n)\ (\exists\, m) \quad m > n\text{”}$$

means "For each natural n, there is a natural m such that $m > n$." We give some more examples.

Let $\mathcal{M} = \{A_i \mid i \in I\}$ be an indexed set family. By definition, $x \in \bigcup A_i$ means that x is in *at least one* of the sets A_i, $i \in I$. In other words, *there is at least one index* $i \in I$ *such that* $x \in A_i$; in symbols,

$$(\exists\, i \in I) \quad x \in A_i.$$

Thus we note that

$$x \in \bigcup_{i \in I} A_i \quad \text{iff} \quad [(\exists\, i \in I)\ x \in A_i].$$

Similarly,

$$x \in \bigcap_{i} A_i \quad \text{iff} \quad [(\forall\, i \in I)\ x \in A_i].$$

Also note that $x \notin \bigcup A_i$ iff x is in *none* of the A_i, i.e.,

$$(\forall\, i) \quad x \notin A_i.$$

Similarly, $x \notin \bigcap A_i$ iff x fails to be in *some* A_i, i.e.,

$$(\exists\, i) \quad x \notin A_i. \quad \text{(Why?)}$$

We now use these remarks to prove Theorem 2(i). We have to show that $S - \bigcup A_i$ has the same elements as $\bigcap(S - A_i)$, i.e., that $x \in S - \bigcup A_i$ iff $x \in \bigcap(S - A_i)$. But, by our definitions, we have

$$\begin{aligned} x \in S - \bigcup A_i &\iff [x \in S,\ x \notin \bigcup A_i] \\ &\iff (\forall\, i)\ [x \in S,\ x \notin A_i] \\ &\iff (\forall\, i)\ x \in S - A_i \\ &\iff x \in \bigcap (S - A_i), \end{aligned}$$

as required.

One proves part (ii) of Theorem 2 quite similarly. (Exercise!)

We shall now dwell on quantifiers more closely. Sometimes a formula $P(x)$ holds not for all $x \in A$, but only for those with an additional property $Q(x)$. This will be written as

$$(\forall\, x \in A \mid Q(x)) \quad P(x),$$

where the vertical stroke stands for "such that." For example, if N is again the naturals, then the formula

$$(\forall x \in N \mid x > 3) \quad x \geq 4 \tag{1}$$

means "for each $x \in N$ *such that* $x > 3$, it is true that $x \geq 4$." In other words, for naturals, $x > 3 \Longrightarrow x \geq 4$ (the arrow stands for "implies"). Thus (1) can also be written as

$$(\forall x \in N) \quad x > 3 \Longrightarrow x \geq 4.$$

In mathematics, we often have to form the *negation* of a formula that starts with one or several quantifiers. It is noteworthy, then, that *each universal quantifier is replaced by an existential one* (*and vice versa*), followed by the negation of the subsequent part of the formula. For example, in calculus, a real number p is called the *limit* of a sequence $x_1, x_2, \ldots, x_n, \ldots$ iff the following is true:

> For every real $\varepsilon > 0$, there is a natural k (depending on ε) such that, for all natural $n > k$, we have $|x_n - p| < \varepsilon$.

If we agree that lower case letters (possibly with subscripts) denote real numbers, and that n, k denote naturals (n, $k \in N$), this sentence can be written as

$$(\forall \varepsilon > 0)\ (\exists k)\ (\forall n > k) \quad |x_n - p| < \varepsilon. \tag{2}$$

Here the expressions "$(\forall \varepsilon > 0)$" and "$(\forall n > k)$" stand for "$(\forall \varepsilon \mid \varepsilon > 0)$" and "$(\forall n \mid n > k)$", respectively (such self-explanatory abbreviations will also be used in other similar cases).

Now, since (2) states that "for all $\varepsilon > 0$" something (i.e., the rest of (2)) is true, the negation of (2) starts with "*there is an* $\varepsilon > 0$" (for which the rest of the formula *fails*). Thus we start with "$(\exists \varepsilon > 0)$", and form the negation of what follows, i.e., of

$$(\exists k)\ (\forall n > k) \quad |x_n - p| < \varepsilon.$$

This negation, in turn, starts with "$(\forall k)$", etc. Step by step, we finally arrive at

$$(\exists \varepsilon > 0)\ (\forall k)\ (\exists n > k) \quad |x_n - p| \geq \varepsilon.$$

Note that here *the choice of* $n > k$ *may depend on* k. To stress it, we often write n_k for n. Thus the negation of (2) finally emerges as

$$(\exists \varepsilon > 0)\ (\forall k)\ (\exists n_k > k) \quad |x_{n_k} - p| \geq \varepsilon. \tag{3}$$

The *order* in which the quantifiers follow each other is *essential*. For example, the formula

$$(\forall n \in N)\ (\exists m \in N) \quad m > n$$

("each $n \in N$ is exceeded by some $m \in N$") is true, but

$$(\exists\, m \in N)\ (\forall\, n \in N) \quad m > n$$

is false. However, two *consecutive* universal quantifiers (or two *consecutive* existential ones) may be interchanged. We briefly write

$$\text{“}(\forall\, x,\, y \in A)\text{” for “}(\forall\, x \in A)\ (\forall\, y \in A),\text{”}$$

and

$$\text{“}(\exists\, x,\, y \in A)\text{” for “}(\exists\, x \in A)\ (\exists\, y \in A),\text{” etc.}$$

We conclude with an important remark. The *universal* quantifier in a formula

$$(\forall\, x \in A) \quad P(x)$$

does not imply the existence of an x for which $P(x)$ is true. It is only meant to imply that *there is no x in A for which $P(x)$ fails*.

The latter is true even if $A = \emptyset$; we then say that "$(\forall\, x \in A)\ P(x)$" is *vacuously true*. For example, the formula $\emptyset \subseteq B$, i.e.,

$$(\forall\, x \in \emptyset) \quad x \in B,$$

is always true (*vacuously*).

Problems in Set Theory

1. Prove Theorem 1 (show that x is in the left-hand set iff it is in the right-hand set). For example, for (d),

$$\begin{aligned} x \in (A \cup B) \cap C &\iff [x \in (A \cup B) \text{ and } x \in C] \\ &\iff [(x \in A \text{ or } x \in B),\ \text{ and } x \in C] \\ &\iff [(x \in A,\, x \in C) \text{ or } (x \in B,\, x \in C)]. \end{aligned}$$

2. Prove that

(i) $-(-A) = A$;

(ii) $A \subseteq B$ iff $-B \subseteq -A$.

3. Prove that

$$A - B = A \cap (-B) = (-B) - (-A) = -[(-A) \cup B].$$

Also, give three expressions for $A \cap B$ and $A \cup B$, in terms of complements.

4. Prove the second duality law (Theorem 2(ii)).

5. Describe geometrically the following sets on the real line:

(i) $\{x \mid x < 0\}$; (ii) $\{x \mid |x| < 1\}$;
(iii) $\{x \mid |x - a| < \varepsilon\}$; (iv) $\{x \mid a < x \le b\}$;
(v) $\{x \mid |x| < 0\}$.

6. Let (a, b) denote the set

$$\{\{a\}, \{a, b\}\}$$

(Kuratowski's definition of an ordered pair).

(i) Which of the following statements are true?

(a) $a \in (a, b)$; (b) $\{a\} \in (a, b)$;
(c) $(a, a) = \{a\}$; (d) $b \in (a, b)$;
(e) $\{b\} \in (a, b)$; (f) $\{a, b\} \in (a, b)$.

(ii) Prove that $(a, b) = (u, v)$ *iff* $a = u$ *and* $b = v$.
[Hint: Consider separately the two cases $a = b$ and $a \neq b$, noting that $\{a, a\} = \{a\}$. Also note that $\{a\} \neq a$.]

7. Describe geometrically the following sets in the xy-plane.

(i) $\{(x, y) \mid x < y\}$;
(ii) $\{(x, y) \mid x^2 + y^2 < 1\}$;
(iii) $\{(x, y) \mid \max(|x|, |y|) < 1\}$;
(iv) $\{(x, y) \mid y > x^2\}$;
(v) $\{(x, y) \mid |x| + |y| < 4\}$;
(vi) $\{(x, y) \mid (x-2)^2 + (y+5)^2 \le 9\}$;
(vii) $\{(x, y) \mid x = 0\}$;
(viii) $\{(x, y) \mid x^2 - 2xy + y^2 < 0\}$;
(ix) $\{(x, y) \mid x^2 - 2xy + y^2 = 0\}$.

8. Prove that

(i) $(A \cup B) \times C = (A \times C) \cup (B \times C)$;
(ii) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$;
(iii) $(X \times Y) - (X' \times Y') = [(X \cap X') \times (Y - Y')] \cup [(X - X') \times Y]$.

[Hint: In each case, show that an *ordered pair* (x, y) is in the left-hand set iff it is in the right-hand set, treating (x, y) as *one* element of the Cartesian product.]

9. Prove the *distributive laws*

(i) $A \cap \bigcup X_i = \bigcup (A \cap X_i)$;
(ii) $A \cup \bigcap X_i = \bigcap (A \cup X_i)$;

(iii) $\left(\bigcap X_i\right) - A = \bigcap(X_i - A)$;

(iv) $\left(\bigcup X_i\right) - A = \bigcup(X_i - A)$;

(v) $\bigcap X_i \cup \bigcap Y_j = \bigcap_{i,\,j}(X_i \cup Y_j)$;[4]

(vi) $\bigcup X_i \cap \bigcup Y_j = \bigcup_{i,\,j}(X_i \cap Y_j)$.

10. Prove that

(i) $\left(\bigcup A_i\right) \times B = \bigcup(A_i \times B)$;

(ii) $\left(\bigcap A_i\right) \times B = \bigcap(A_i \times B)$;

(iii) $\left(\bigcap_i A_i\right) \times \left(\bigcap_j B_j\right) = \bigcap_{i,j}(A_i \times B_i)$;

(iv) $\left(\bigcup_i A_i\right) \times \left(\bigcup_j B_j\right) = \bigcup_{i,\,j}(A_i \times B_j)$.

§§4–7. Relations. Mappings

In §§1–3, we have already considered *sets of ordered pairs*, such as Cartesian products $A \times B$ or sets of the form $\{(x,\,y) \mid P(x,\,y)\}$ (cf. §§1–3, Problem 7). If the pair $(x,\,y)$ is an element of such a set R, we write

$$(x,\,y) \in R,$$

treating $(x,\,y)$ as *one* thing. Note that this *does not* imply that x and y taken *separately* are members of R (in which case we would write $x,\,y \in R$). We call $x,\,y$ the *terms* of $(x,\,y)$.

In mathematics, it is customary to call any set of ordered pairs a *relation*. For example, all sets listed in Problem 7 of §§1–3 are relations. Since relations are *sets*, equality $R = S$ for relations means that they consist of the same elements (ordered pairs), i.e., that

$$(x,\,y) \in R \Longleftrightarrow (x,\,y) \in S.$$

If $(x,\,y) \in R$, we call y an *R-relative* of x; we also say that y is *R-related to x* or that the *relation R holds between x and y* (in this order). Instead of $(x,\,y) \in R$, we also write xRy, and often replace "R" by special symbols like $<$, $\sim$, etc. Thus, in case (i) of Problem 7 above, "xRy" means that $x < y$.

Replacing all pairs $(x,\,y) \in R$ by the *inverse* pairs $(y,\,x)$, we obtain a new relation, called the *inverse of R* and denoted R^{-1}. Clearly, $xR^{-1}y$ iff yRx; thus

$$R^{-1} = \{(x,\,y) \mid yRx\} = \{(y,\,x) \mid xRy\}.$$

[4] Here we work with *two* set families, $\{X_i \mid i \in I\}$ and $\{Y_j \mid j \in J\}$; similarly in other such cases.

Hence R, in turn, is the inverse of R^{-1}; i.e.,

$$(R^{-1})^{-1} = R.$$

For example, the relations $<$ and $>$ between numbers are inverse to each other; so also are the relations $\subseteq$ and $\supseteq$ between sets. (We may treat "$\subseteq$" as the name of *the set of all pairs* (X, Y) *such that* $X \subseteq Y$ in a given space.)

If R contains the pairs (x, x'), (y, y'), (z, z'), ..., we shall write

$$R = \begin{pmatrix} x & y & z & \cdots \\ x' & y' & z' & \end{pmatrix}; \text{ e.g., } R = \begin{pmatrix} 1 & 4 & 1 & 3 \\ 2 & 2 & 1 & 1 \end{pmatrix}. \tag{1}$$

To obtain R^{-1}, we simply interchange the upper and lower rows in (1).

Definition 1.

The set of all *left* terms x of pairs $(x, y) \in R$ is called the *domain* of R, denoted D_R. The set of all *right* terms of these pairs is called the *range* of R, denoted D'_R. Clearly, $x \in D_R$ iff xRy for some y. In symbols,

$$x \in D_R \iff (\exists\, y)\ xRy; \text{ similarly, } y \in D'_R \iff (\exists\, x)\ xRy.$$

In (1), D_R is the upper row, and D'_R is the lower row. Clearly,

$$D_{R^{-1}} = D'_R \text{ and } D'_{R^{-1}} = D_R.$$

For example, if

$$R = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 2 & 1 \end{pmatrix},$$

then

$$D_R = D'_{R^{-1}} = \{1, 4\} \text{ and } D'_R = D_{R^{-1}} = \{1, 2\}.$$

Definition 2.

The *image* of a set A under a relation R (briefly, the R-*image* of A) is the set of all R-relatives of elements of A, denoted $R[A]$. The *inverse image* of A under R is the image of A under the *inverse* relation, i.e., $R^{-1}[A]$. If A consists of a single element, $A = \{x\}$, then $R[A]$ and $R^{-1}[A]$ are also written $R[x]$ and $R^{-1}[x]$, respectively, instead of $R[\{x\}]$ and $R^{-1}[\{x\}]$.

Example.

Let

$$R = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 7 \\ 1 & 3 & 4 & 5 & 3 & 4 & 1 & 3 & 5 & 1 \end{pmatrix},\ A = \{1, 2\},\ B = \{2, 4\}.$$

Then

$R[1] = \{1, 3, 4\}$;	$R[2] = \{3, 5\}$;	$R[3] = \{1, 3, 4, 5\}$
$R[5] = \emptyset$;	$R^{-1}[1] = \{1, 3, 7\}$;	$R^{-1}[2] = \emptyset$;
$R^{-1}[3] = \{1, 2, 3\}$;	$R^{-1}[4] = \{1, 3\}$;	$R[A] = \{1, 3, 4, 5\}$;
$R^{-1}[A] = \{1, 3, 7\}$;	$R[B] = \{3, 5\}$.	

By definition, $R[x]$ is the set of all R-relatives of x. Thus

$$y \in R[x] \quad \text{iff} \quad (x, y) \in R; \text{ i.e., } xRy.$$

More generally, $y \in R[A]$ means that $(x, y) \in R$ *for some* $x \in A$. In symbols,

$$y \in R[A] \Longleftrightarrow (\exists\, x \in A)\ (x, y) \in R.$$

Note that $R[A]$ is *always* defined.

We shall now consider an especially important kind of relation.

Definition 3.

A relation R is called a *mapping* (*map*), or a *function*, or a *transformation*, iff every element $x \in D_R$ has a *unique* R-relative, so that $R[x]$ consists of a *single* element. This unique element is denoted by $R(x)$ and is called the *function value* at x (under R). Thus $R(x)$ is the only member of $R[x]$.[1]

If, in addition, different elements of D_R have *different* images, R is called a *one-to-one* (or *one-one*) map. In this case,

$$x \neq y\ (x, y \in D_R) \text{ implies } R(x) \neq R(y);$$

equivalently,

$$R(x) = R(y) \text{ implies } x = y.$$

In other words, no two pairs belonging to R have the same left, or the same right, terms. This shows that *R is one to one iff R^{-1}, too, is a map*.[2] Mappings are often denoted by the letters f, g, h, F, ψ, etc.

[1] Equivalently, R is a map iff $(x, y) \in R$ and $(x, z) \in R$ implies that $y = z$. (Why?)

[2] Note that R^{-1} always exists as a *relation*, but it need not be a *map*. For example,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 8 \end{pmatrix}$$

is a map, but

$$f^{-1} = \begin{pmatrix} 2 & 3 & 3 & 8 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

is *not*. (Why?) Here f is *not* one to one.

A mapping f is said to be "*from A to B*" iff $D_f = A$ and $D'_f \subseteq B$; we then write

$$f\colon A \to B \quad (\text{“}f \text{ maps } A \text{ into } B\text{”}).$$

If, in particular, $D_f = A$ and $D'_f = B$, we call f a map of A *onto* B, and we write

$$f\colon A \underset{\text{onto}}{\longrightarrow} B \quad (\text{“}f \text{ maps } A \textit{ onto } B\text{”}).$$

If f is both onto and one to one, we write

$$f\colon A \underset{\text{onto}}{\longleftrightarrow} B$$

($f\colon A \longleftrightarrow B$ means that f is one to one).

All pairs belonging to a mapping f have the form $(x, f(x))$ where $f(x)$ is the function value at x, i.e., the unique f-relative of x, $x \in D_f$. Therefore, in order to define some function f, *it suffices to specify its domain D_f and the function value $f(x)$ for each $x \in D_f$.* We shall often use such definitions. It is customary to say that f *is defined on* A (or "f is a function *on* A") iff $A = D_f$.

Examples.

(a) The relation

$$R = \{(x, y) \mid x \text{ is the wife of } y\}$$

is a one-to-one map of the set of all wives onto the set of all husbands. R^{-1} is here a one-to-one map of the set of of all husbands ($= D'_R$) onto the set of all wives ($= D_R$).

(b) The relation

$$f = \{(x, y) \mid y \text{ is the father of } x\}$$

is a map of the set of all people onto the set of their fathers. It is not one to one since several persons may have the same father (f-relative), and so $x \neq x'$ does not imply $f(x) \neq f(x')$.

(c) Let

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 8 \end{pmatrix}.$$

Then g is a map of $D_g = \{1, 2, 3, 4\}$ onto $D'_g = \{2, 3, 8\}$, with

$$g(1) = 2,\ g(2) = 2,\ g(3) = 3,\ g(4) = 8.$$

(As noted above, these formulas may serve to *define* g.) It is not one to one since $g(1) = g(2)$, so g^{-1} is not a map.

(d) Consider

$$f\colon N \to N, \text{ with } f(x) = 2x \text{ for each } x \in N.$$[3]

By what was said above, f is well defined. It is one to one since $x \neq y$ implies $2x \neq 2y$. Here $D_f = N$ (the naturals), but D'_f consists of *even* naturals only. Thus f is not *onto* N (it is onto a *smaller* set, the *even* naturals); f^{-1} maps the even naturals onto all of N.

The domain and range of a relation may be quite arbitrary sets. In particular, we can consider functions f in which each element of the domain D_f is itself an ordered pair (x, y) or n-tuple $(x_1, x_2, \ldots, x_n)$. Such mappings are called *functions of two* (*respectively*, n) *variables*. To any n-tuple $(x_1, \ldots, x_n)$ that belongs to D_f, the function f assigns a unique function value, denoted by $f(x_1, \ldots, x_n)$. It is convenient to regard $x_1, x_2, \ldots, x_n$ as certain variables; then the function value, too, becomes a variable depending on the $x_1, \ldots, x_n$. Often D_f consists of all ordered n-tuples of elements taken from a set A, i.e., $D_f = A^n$ (cross-product of n sets, each equal to A). The *range* may be an arbitrary set B; so $f\colon A^n \to B$. Similarly, $f\colon A \times B \to C$ is a function of *two* variables, with $D_f = A \times B$, $D'_f \subseteq C$.

Functions of two variables are also called (*binary*) *operations*. For example, addition of natural numbers may be treated as a map $f\colon N \times N \to N$, with $f(x, y) = x + y$.

Definition 4.

A relation R is said to be

(i) *reflexive* iff we have xRx for each $x \in D_R$;

(ii) *symmetric* iff xRy always implies yRx;

(iii) *transitive* iff xRy combined with yRz always implies xRz.

R is called an *equivalence relation* on a set A iff $A = D_R$ and R has all the three properties (i), (ii), and (iii). For example, such is the *equality relation* on A (also called the *identity map* on A) denoted

$$I_A = \{(x, y) \mid x \in A,\ x = y\}.$$

Equivalence relations are often denoted by special symbols resembling equality, such as $\equiv$, $\approx$, $\sim$, etc. The formula xRy, where R is such a symbol, is read

"*x is equivalent* (or *R-equivalent*) *to y*,"

[3] This is often abbreviated by saying "consider *the function* $f(x) = 2x$ on N." However, one should remember that $f(x)$ is actually not the *function* f (a set of ordered pairs) but only a single element of the range of f. A better expression is "*f is the map $x \to 2x$ on N*" or "*f carries x into $2x$* ($x \in N$)."

and $R[x]=\{y \mid xRy\}$ (i.e., the R-image of x) is called the *R-equivalence class* (briefly *R-class*) of x in A; it consists of all elements that are R-equivalent to x and *hence to each other* (for xRy and xRz imply first yRx, by symmetry, and hence yRz, by transitivity). Each such element is called a *representative* of the given R-class, or its *generator*. We often write $[x]$ for $R[x]$.

Examples.

(a′) The inequality relation $<$ between real numbers is transitive since

$$x<y \text{ and } y<z \text{ implies } x<z;$$

it is neither reflexive nor symmetric. (Why?)

(b′) The inclusion relation $\subseteq$ between sets is reflexive (for $A \subseteq A$) and transitive (for $A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$), but it is not symmetric.

(c′) The membership relation $\in$ between an element and a set is neither reflexive nor symmetric nor transitive ($x \in A$ and $A \in \mathcal{M}$ *does not* imply $x \in \mathcal{M}$).

(d′) Let R be the *parallelism* relation between lines in a plane, i.e., the set of all pairs (X, Y), where X and Y are parallel lines. Writing $\parallel$ for R, we have $X \parallel X$, $X \parallel Y$ implies $Y \parallel X$, and ($X \parallel Y$ and $Y \parallel Z$) implies $X \parallel Z$, so R is an equivalence relation. An R-class here consists of all lines parallel to a given line in the plane.

(e′) Congruence of triangles is an equivalence relation. (Why?)

Theorem 1. *If R (also written $\equiv$) is an equivalence relation on A, then all R-classes are disjoint from each other, and A is their union.*

Proof. Take two R-classes, $[p] \neq [q]$. Seeking a contradiction, suppose they are *not* disjoint, so

$$(\exists\, x) \quad x \in [p] \text{ and } x \in [q];$$

i.e., $p \equiv x \equiv q$ and hence $p \equiv q$. But then, by symmetry and transitivity,

$$y \in [p] \Leftrightarrow y \equiv p \Leftrightarrow y \equiv q \Leftrightarrow y \in [q];$$

i.e., $[p]$ and $[q]$ *consist of the same elements* y, contrary to assumption $[p] \neq [q]$. Thus, indeed, any two (distinct) R-classes are disjoint.

Also, by reflexivity,

$$(\forall\, x \in A) \quad x \equiv x,$$

i.e., $x \in [x]$. Thus each $x \in A$ is in some R-class (namely, in $[x]$); so all of A is in the *union* of such classes,

$$A \subseteq \bigcup_x R[x].$$

Conversely,

$$(\forall\, x) \quad R[x] \subseteq A$$

since

$$y \in R[x] \Rightarrow xRy \Rightarrow yRx \Rightarrow (y,\, x) \in R \Rightarrow y \in D_R = A,$$

by definition. Thus A contains all $R[x]$, hence their union, and so

$$A = \bigcup_x R[x]. \quad \square$$

Problems on Relations and Mappings

1. For the relations specified in Problem 7 of §§1–3, find D_R, D'_R, and R^{-1}. Also, find $R[A]$ and $R^{-1}[A]$ if

(a) $A = \{\frac{1}{2}\}$; (b) $A = \{1\}$;
(c) $A = \{0\}$; (d) $A = \emptyset$;
(e) $A = \{0,\, 3,\, -15\}$; (f) $A = \{3,\, 4,\, 7,\, 0,\, -1,\, 6\}$;
(g) $A = \{x \mid -20 < x < 5\}$.

2. Prove that if $A \subseteq B$, then $R[A] \subseteq R[B]$. Disprove the converse by a counterexample.

3. Prove that

(i) $R[A \cup B] = R[A] \cup R[B]$;

(ii) $R[A \cap B] \subseteq R[A] \cap R[B]$;

(iii) $R[A - B] \supseteq R[A] - R[B]$.

Disprove reverse inclusions in (ii) and (iii) by examples. Do (i) and (ii) with A, B replaced by an arbitrary set family $\{A_i \mid i \in I\}$.

4. Under which conditions are the following statements true?

(i) $R[x] = \emptyset$; (ii) $R^{-1}[x] = \emptyset$;
(iii) $R[A] = \emptyset$; (iv) $R^{-1}[A] = \emptyset$.

5. Let $f\colon N \to N$ ($N = \{\text{naturals}\}$). For each of the following functions, specify $f[N]$, i.e., D'_f, and determine whether f is one to one and onto N, given that for all $x \in N$,

(i) $f(x) = x^3$; (ii) $f(x) = 1$; (iii) $f(x) = |x| + 3$;
(iv) $f(x) = x^2$; (v) $f(x) = 4x + 5$.

Do all this also if N denotes

(a) the set of *all* integers;

(b) the set of all reals.

6. Prove that for any *mapping* f and any sets A, B, A_i $(i \in I)$,

(a) $f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B]$;

(b) $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$;

(c) $f^{-1}[A - B] = f^{-1}[A] - f^{-1}[B]$;

(d) $f^{-1}[\bigcup_i A_i] = \bigcup_i f^{-1}[A_i]$;

(e) $f^{-1}[\bigcap_i A_i] = \bigcap_i f^{-1}[A_i]$.

Compare with Problem 3.

[Hint: First verify that $x \in f^{-1}[A]$ iff $x \in D_f$ and $f(x) \in A$.]

7. Let f be a map. Prove that

(a) $f[f^{-1}[A]] \subseteq A$;

(b) $f[f^{-1}[A]] = A$ if $A \subseteq D'_f$;

(c) if $A \subseteq D_f$ and f is one to one, $A = f^{-1}[f[A]]$.

Is $f[A] \cap B \subseteq f[A \cap f^{-1}[B]]$?

8. Is R an equivalence relation on the set J of all integers, and, if so, what are the R-classes, if

(a) $R = \{(x, y) \mid x - y \text{ is divisible by a fixed } n\}$;

(b) $R = \{(x, y) \mid x - y \text{ is } \textit{odd}\}$;

(c) $R = \{(x, y) \mid x - y \text{ is a prime}\}$.

(x, y, n denote *integers*.)

9. Is any relation in Problem 7 of §§1–3 reflexive? Symmetric? Transitive?

10. Show by examples that R may be

(a) reflexive and symmetric, without being transitive;

(b) reflexive and transitive without being symmetric.

Does symmetry plus transitivity imply reflexivity? Give a proof or counterexample.

§8. Sequences[1]

By an *infinite sequence* (briefly *sequence*) we mean a mapping (call it u) whose domain is N (all natural numbers 1, 2, 3, ...); D_u may also contain 0.

[1] This section may be deferred until Chapter 2, §13.

A *finite sequence* is a map u in which D_u consists of all positive (or non-negative) integers *less than a fixed integer* p. The *range* D'_u of any sequence u may be an arbitrary set B; we then call u a sequence *of elements of* B, or *in* B. For example,

$$u = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n & \dots \\ 2 & 4 & 6 & 8 & \dots & 2n & \dots \end{pmatrix} \tag{1}$$

is a sequence with

$$D_u = N = \{1, 2, 3, \dots\}$$

and with function values

$$u(1) = 2,\ u(2) = 4,\ u(n) = 2n, \quad n = 1, 2, 3, \dots.$$

Instead of $u(n)$ we usually write u_n ("index notation"), and call u_n the *nth term* of the sequence. If n is treated as a *variable*, u_n is called the *general term* of the sequence, and $\{u_n\}$ is used to denote the entire (infinite) sequence, as well as its range D'_u (whichever is meant, will be clear from the context). The formula $\{u_n\} \subseteq B$ means that $D'_u \subseteq B$, i.e., that u is a sequence in B. To determine a sequence, it suffices to define its general term u_n by some formula or rule.[2] In (1) above, $u_n = 2n$.

Often we omit the mention of $D_u = N$ (since it is *known*) and give only the range D'_u. Thus instead of (1), we briefly write

$$2, 4, 6, \dots, 2n, \dots$$

or, more generally,

$$u_1, u_2, \dots, u_n, \dots.$$

Yet it should be remembered that u is a set of *pairs* (a map).

If all u_n are *distinct* (different from each other), u is a *one-to-one* map. However, this need not be the case. It may even occur that all u_n are equal (then u is said to be constant); e.g., $u_n = 1$ yields the sequence 1, 1, 1, …, 1, …, i.e.,

$$u = \begin{pmatrix} 1 & 2 & 3 & \dots & n & \dots \\ 1 & 1 & 1 & \dots & 1 & \dots \end{pmatrix}. \tag{2}$$

Note that here u is an *infinite* sequence (since $D_u = N$), even though its range D'_u has only one element, $D'_u = \{1\}$. (In *sets*, repeated terms count as *one* element; but the *sequence* u consists of infinitely many distinct *pairs* $(n, 1)$.) If all u_n are real numbers, we call u a *real sequence*. For such sequences, we have the following definitions.

[2] However, such a formula may not exist; the u_n may even be chosen "at random."

Definition 1.

A real sequence $\{u_n\}$ is said to be *monotone* (or *monotonic*) iff it is either *nondecreasing*, i.e.,

$$(\forall\, n) \quad u_n \le u_{n+1},$$

or *nonincreasing*, i.e.,

$$(\forall\, n) \quad u_n \ge u_{n+1}.$$

Notation: $\{u_n\}\uparrow$ and $\{u_n\}\downarrow$, respectively. If instead we have the *strict* inequalities $u_n < u_{n+1}$ (respectively, $u_n > u_{n+1}$), we call $\{u_n\}$ *strictly monotone* (increasing or decreasing).

A similar definition applies to sequences of *sets*.

Definition 2.

A sequence of sets $A_1, A_2, \ldots, A_n, \ldots$ is said to be *monotone* iff it is either *expanding*, i.e.,

$$(\forall\, n) \quad A_n \subseteq A_{n+1},$$

or *contracting*, i.e.,

$$(\forall\, n) \quad A_n \supseteq A_{n+1}.$$

Notation: $\{A_n\}\uparrow$ and $\{A_n\}\downarrow$, respectively. For example, any sequence of concentric solid spheres (treated as *sets of points*), with increasing radii, is expanding; if the radii decrease, we obtain a contracting sequence.

Definition 3.

Let $\{u_n\}$ be *any* sequence, and let

$$n_1 < n_2 < \cdots < n_k < \cdots$$

be a *strictly increasing* sequence of natural numbers. Select from $\{u_n\}$ those terms whose subscripts are $n_1, n_2, \ldots, n_k, \ldots$. Then the sequence $\{u_{n_k}\}$ so selected (with kth term equal to u_{n_k}), is called the *subsequence* of $\{u_n\}$, determined by the subscripts n_k, $k = 1, 2, 3, \ldots$.

Thus (roughly) a subsequence is any sequence obtained from $\{u_n\}$ by dropping some terms, *without changing the order of the remaining terms* (this is ensured by the inequalities $n_1 < n_2 < \cdots < n_k < \cdots$ where the n_k are the subscripts of the *remaining* terms). For example, let us select from (1) the subsequence of terms whose subscripts are *primes* (including 1). Then the subsequence is

$$2, 4, 6, 10, 14, 22, \ldots,$$

i.e.,

$$u_1, u_2, u_3, u_5, u_7, u_{11}, \ldots.$$

All these definitions apply to finite sequences accordingly. Observe that every sequence arises by "numbering" the elements of its range (the terms): u_1 is the *first* term, u_2 is the *second* term, and so on. By so numbering, we put the terms in a certain *order*, determined by their subscripts 1, 2, 3, ... (like the numbering of buildings in a street, of books in a library, etc.). The question now arises: Given a set A, is it always possible to "number" its elements *by integers*? As we shall see in §9, this is not always the case. This leads us to the following definition.

Definition 4.

A set A is said to be *countable* iff A is contained in the range of some sequence (briefly, *the elements of A can be put in a sequence*).

If, in particular, this sequence can be chosen finite, we call A a *finite* set. (The empty set is finite.)

Sets that are not finite are said to be *infinite*.

Sets that are not countable are said to be *uncountable*.

Note that all finite sets are countable. The simplest example of an infinite countable set is $N = \{1, 2, 3, \ldots\}$.

§9. Some Theorems on Countable Sets[1]

We now derive some corollaries of Definition 4 in §8.

Corollary 1. *If a set A is countable or finite, so is any subset $B \subseteq A$.*

For if $A \subset D'_u$ for a sequence u, then certainly $B \subseteq A \subseteq D'_u$.

Corollary 2. *If A is uncountable (or just infinite), so is any superset $B \supseteq A$.*

For, if B were countable or finite, so would be $A \subseteq B$, by Corollary 1.

Theorem 1. *If A and B are countable, so is their cross product $A \times B$.*

Proof. If A or B is $\emptyset$, then $A \times B = \emptyset$, and there is nothing to prove.

Thus let A and B be nonvoid and countable. We may assume that they *fill* two infinite sequences, $A = \{a_n\}$, $B = \{b_n\}$ (*repeat* terms if necessary). Then, by definition, $A \times B$ is the set of all ordered pairs of the form

$$(a_n, b_m), \quad n, m \in N.$$

Call $n + m$ the *rank* of the pair (a_n, b_m). For each $r \in N$, there are $r - 1$ pairs of rank r:

$$(a_1, b_{r-1}), \ (a_2, b_{r-2}), \ \ldots, \ (a_{r-1}, b_1). \tag{1}$$

[1] This section may be deferred until Chapter 5, §4.

We now put all pairs (a_n, b_m) in *one* sequence as follows. We start with

$$(a_1, b_1)$$

as the first term; then take the two pairs of rank three,

$$(a_1, b_2),\ (a_2, b_1);$$

then the three pairs of rank four, and so on. At the $(r-1)$st step, we take all pairs of rank r, in the order indicated in (1).

Repeating this process for all ranks ad infinitum, we obtain the sequence of pairs

$$(a_1, b_1),\ (a_1, b_2),\ (a_2, b_1),\ (a_1, b_3),\ (a_2, b_2),\ (a_3, b_1),\ \ldots,$$

in which $u_1 = (a_1, b_1)$, $u_2 = (a_1, b_2)$, etc.

By construction, this sequence contains *all pairs of all ranks* r, hence all pairs that form the set $A \times B$ (for every such pair has some rank r and so it must eventually occur in the sequence). Thus $A \times B$ *can* be put in a sequence. □

Corollary 3. *The set* R *of all* rational *numbers*[2] *is countable.*

Proof. Consider first the set Q of all *positive* rationals, i.e.,

$$\text{fractions } \frac{n}{m}, \text{ with } n, m \in N.$$

We may formally identify them with ordered pairs (n, m), i.e., with $N \times N$. We call $n + m$ the *rank* of (n, m). As in Theorem 1, we obtain the sequence

$$\frac{1}{1},\ \frac{1}{2},\ \frac{2}{1},\ \frac{1}{3},\ \frac{2}{2},\ \frac{3}{1},\ \frac{1}{4},\ \frac{2}{3},\ \frac{3}{2},\ \frac{4}{1},\ \ldots.$$

By dropping reducible fractions and inserting also 0 and the negative rationals, we put R into the sequence

$$0,\ 1,\ -1,\ \frac{1}{2},\ -\frac{1}{2},\ 2,\ -2,\ \frac{1}{3},\ -\frac{1}{3},\ 3,\ -3,\ \ldots, \text{ as required.} \quad \square$$

Theorem 2. *The union of any sequence* $\{A_n\}$ *of countable sets is countable.*

Proof. As each A_n is countable, we may put

$$A_n = \{a_{n1}, a_{n2}, \ldots, a_{nm}, \ldots\}.$$

(The double subscripts are to distinguish the sequences representing different sets A_n.) As before, we may assume that all sequences are infinite. Now, $\bigcup_n A_n$ obviously consists of the elements of *all* A_n *combined*, i.e., *all* a_{nm} $(n, m \in N)$. We call $n + m$ the *rank* of a_{nm} and proceed as in Theorem 1, thus obtaining

$$\bigcup_n A_n = \{a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \ldots\}.$$

[2] A number is *rational* iff it is the ratio of two *integers*, p/q, $q \neq 0$.

Thus $\bigcup_n A_n$ can be put in a sequence. □

Note 1. Theorem 2 is briefly expressed as

"*Any countable union of countable sets is a countable set.*"

(The term "*countable union*" means "union of a *countable* family of sets", i.e., a family of sets whose elements can be put in a sequence $\{A_n\}$.) In particular, if A and B are countable, so are $A \cup B$, $A \cap B$, and $A - B$ (by Corollary 1).

Note 2. From the proof it also follows that *the range of any double sequence* $\{a_{nm}\}$ *is countable.* (A *double sequence* is a function u whose domain D_u is $N \times N$; say, $u\colon N \times N \to B$. If $n, m \in N$, we write u_{nm} for $u(n, m)$; here $u_{nm} = a_{nm}$.)

To prove the existence of *uncountable* sets, we shall now show that the interval

$$[0, 1) = \{x \mid 0 \leq x < 1\}$$

of the real axis is uncountable.

We assume as known the fact that each real number $x \in [0, 1)$ has a unique infinite decimal expansion

$$0.x_1, x_2, \ldots, x_n, \ldots,$$

where the x_n are the decimal digits (possibly zeros), and the sequence $\{x_n\}$ does not terminate in *nines* (this ensures *uniqueness*).[3]

Theorem 3. *The interval* $[0, 1)$ *of the real axis is uncountable.*

Proof. We must show that no sequence can comprise *all* of $[0, 1)$. Indeed, given any $\{u_n\}$, write each term u_n as an infinite decimal fraction; say,

$$u_n = 0.a_{n1}, a_{n2}, \ldots, a_{nm}, \ldots.$$

Next, construct a *new* decimal fraction

$$z = 0.x_1, x_2, \ldots, x_n, \ldots,$$

choosing its digits x_n as follows.

If a_{nn} (i.e., the nth digit of u_n) is 0, put $x_n = 1$; if, however, $a_{nn} \neq 0$, put $x_n = 0$. Thus, in all cases, $x_n \neq a_{nn}$, i.e., *z differs from each u_n in at least one decimal digit* (namely, the nth digit). It follows that z is different from all u_n and hence is not in $\{u_n\}$, even though $z \in [0, 1)$.

Thus, no matter what the choice of $\{u_n\}$ was, we found some $z \in [0, 1)$ *not* in the range of that sequence. Hence *no* $\{u_n\}$ contains all of $[0, 1)$. □

Note 3. By Corollary 2, any superset of $[0, 1)$, e.g., the entire real axis, is *uncountable*. See also Problem 4 below.

[3] For example, instead of 0.49999 . . . , we write 0.50000 . . . .

Note 4. Observe that the numbers a_{nn} used in the proof of Theorem 3 form the *diagonal* of the infinitely extending square composed of all a_{nm}. Therefore, the method used above is called the *diagonal process* (due to G. Cantor).

Problems on Countable and Uncountable Sets

1. Prove that if A is countable but B is not, then $B - A$ is uncountable.
[Hint: If $B - A$ were countable, so would be

$$(B - A) \cup A \supseteq B. \quad \text{(Why?)}$$

Use Corollary 1.]

2. Let f be a mapping, and $A \subseteq D_f$. Prove that
(i) if A is countable, so is $f[A]$;
(ii) if f is one to one and A is uncountable, so is $f[A]$.

[Hints: (i) If $A = \{u_n\}$, then

$$f[A] = \{f(u_1), f(u_2), \ldots, f(u_n), \ldots\}.$$

(ii) If $f[A]$ were countable, so would be $f^{-1}[f[A]]$, by (i). Verify that

$$f^{-1}[f[A]] = A$$

here; cf. Problem 7 in §§4–7.]

3. Let a, b be real numbers ($a < b$). Define a map f on $[0, 1)$ by

$$f(x) = a + x(b - a).$$

Show that f is one to one and *onto* the interval $[a, b) = \{x \mid a \leq x < b\}$. From Problem 2, deduce that $[a, b)$ is uncountable. Hence, by Problem 1, *so is* $(a, b) = \{x \mid a < x < b\}$.

4. Show that between any real numbers a, b ($a < b$) there are *uncountably many irrationals*, i.e., numbers that are not rational.
[Hint: By Corollary 3 and Problems 1 and 3, the set $(a, b) - R$ is uncountable. Explain in detail.]

5. Show that every infinite set A contains a *countably infinite* set, i.e., an infinite sequence of distinct terms.
[Hint: Fix any $a_1 \in A$; A cannot consist of a_1 alone, so there is another element

$$a_2 \in A - \{a_1\}. \quad \text{(Why?)}$$

Again, $A \neq \{a_1, a_2\}$, so there is an $a_3 \in A - \{a_1, a_2\}$. (Why?) Continue thusly ad infinitum to obtain the required sequence $\{a_n\}$. Why are all a_n *distinct*?]

***6.** From Problem 5, prove that if A is infinite, there is a map $f\colon A \to A$ that is one to one but not *onto* A.
[Hint: With a_n as in Problem 5, define $f(a_n) = a_{n+1}$. If, however, x is none of the a_n, put $f(x) = x$. Observe that $f(x) = a_1$ is never true, so f is not *onto* A. Show, however, that f is one to one.]

***7.** Conversely (cf. Problem 6), prove that if there is a map $f\colon A \to A$ that is one to one but *not onto* A, then A contains an infinite sequence $\{a_n\}$ of distinct terms.

[Hint: As f is not *onto* A, there is $a_1 \in A$ such that $a_1 \notin f[A]$. (Why?) Fix a_1 and define

$$a_2 = f(a_1),\ a_3 = f(a_2),\ \ldots,\ a_{n+1} = f(a_n),\ \ldots \text{ ad infinitum.}$$

To prove distinctness, show that each a_n is distinct from all a_m with $m > n$. For a_1, this is true since $a_1 \notin f[A]$, whereas $a_m \in f[A]$ $(m > 1)$. Then proceed inductively.]

Chapter 2

Real Numbers. Fields

§§1–4. Axioms and Basic Definitions

Real numbers can be constructed step by step: first the integers, then the rationals, and finally the irrationals.[1] Here, however, we shall assume the set of all real numbers, denoted E^1, as *already given*, without attempting to reduce this notion to simpler concepts. We shall also accept without definition (as *primitive* concepts) the notions of the *sum* $(a + b)$ and the *product*, $(a \cdot b)$ or (ab), of two real numbers, as well as the *inequality relation* $<$ (read "*less than*"). Note that $x \in E^1$ means "*x is in* E^1," i.e., "*x is a real number*."

It is an important fact that all arithmetic properties of reals can be deduced from several simple axioms, listed (and named) below.

Axioms of Addition and Multiplication

I (closure laws). *The sum* $x + y$, *and the product* xy, *of any real numbers are real numbers themselves.* In symbols,

$$(\forall\, x, y \in E^1) \quad (x + y) \in E^1 \text{ and } (xy) \in E^1.$$

II (commutative laws).

$$(\forall\, x, y \in E^1) \quad x + y = y + x \text{ and } xy = yx.$$

III (associative laws).

$$(\forall\, x, y, z \in E^1) \quad (x + y) + z = x + (y + z) \text{ and } (xy)z = x(yz).$$

IV (existence of neutral elements).

(a) *There is a* (*unique*) *real number*, *called zero* (0), *such that*, *for all real* x, $x + 0 = x$.

[1] See the author's *Basic Concepts of Mathematics*, Chapter 2, §15.

(b) *There is a (unique) real number, called one* (1), *such that* $1 \neq 0$ *and, for all real* x, $x \cdot 1 = x$.

In symbols,

(a) $(\exists!\, 0 \in E^1)\ (\forall\, x \in E^1) \quad x + 0 = x;$

(b) $(\exists!\, 1 \in E^1)\ (\forall\, x \in E^1) \quad x \cdot 1 = x,\ 1 \neq 0.$

(The real numbers 0 and 1 are called the *neutral elements* of addition and multiplication, respectively.)

V (existence of inverse elements).

(a) *For every real* x, *there is a (unique) real, denoted* $-x$, *such that* $x + (-x) = 0$.

(b) *For every real* x other than 0, *there is a (unique) real, denoted* x^{-1}, *such that* $x \cdot x^{-1} = 1$.

In symbols,

(a) $(\forall\, x \in E^1)\ (\exists!\, {-x} \in E^1) \quad x + (-x) = 0;$

(b) $(\forall\, x \in E^1 \mid x \neq 0)\ (\exists!\, x^{-1} \in E^1) \quad xx^{-1} = 1.$

(The real numbers $-x$ and x^{-1} are called, respectively, the *additive inverse* (or the *symmetric*) and the *multiplicative inverse* (or the *reciprocal*) of x.)

VI (distributive law).

$$(\forall\, x,\, y,\, z \in E^1) \quad (x + y)z = xz + yz.$$

Axioms of Order

VII (trichotomy). *For any real* x *and* y, *we have*

$$\textit{either } x < y \textit{ or } y < x \textit{ or } x = y$$

but never two of these relations together.

VIII (transitivity).

$$(\forall\, x,\, y,\, z \in E^1) \quad x < y \textit{ and } y < z \textit{ implies } x < z.$$

IX (monotonicity of addition and multiplication). *For any* x, y, $z \in E^1$, *we have*

(a) $x < y$ *implies* $x + z < y + z$;

(b) $x < y$ *and* $z > 0$ *implies* $xz < yz$.

An additional axiom will be stated in §§8–9.

Note 1. The *uniqueness* assertions in Axioms IV and V are actually redundant since they can be deduced from other axioms. We shall not dwell on this.

Note 2. *Zero has no reciprocal*; i.e., *for no x is* $0x = 1$. In fact, $0x = 0$. For, by Axioms VI and IV,

$$0x + 0x = (0 + 0)x = 0x = 0x + 0.$$

Cancelling $0x$ (i.e., adding $-0x$ on both sides), we obtain $0x = 0$, by Axioms III and V(a).

Note 3. Due to Axioms VII and VIII, real numbers may be regarded as given in a certain *order* under which smaller numbers precede the larger ones. (This is why we speak of "axioms of *order*.") The ordering of real numbers can be visualized by "plotting" them as points on a directed line ("the real axis") in a well-known manner. Therefore, E^1 is also often called "*the real axis*," and real numbers are called "*points*"; we say "the *point* x" instead of "the *number* x."

Observe that the axioms only state certain properties of real numbers *without specifying what these numbers are*. Thus we may treat the reals as just *any* mathematical objects satisfying our axioms, but otherwise arbitrary. Indeed, our theory also applies to any other set of objects (numbers or not), provided they satisfy our axioms with respect to a certain relation of order ($<$) and certain operations ($+$) and ($\cdot$), which may, but need not, be ordinary addition and multiplication. Such sets exist indeed. We now give them a name.

Definition 1.

A *field* is any set F of objects, with two operations ($+$) and ($\cdot$) defined in it in such a manner that they satisfy Axioms I–VI listed above (with E^1 replaced by F, of course).

If F is also endowed with a relation $<$ satisfying Axioms VII to IX, we call F an *ordered field*.

In this connection, postulates I to IX are called *axioms of an* (*ordered*) *field*.

By Definition 1, E^1 is an ordered field. Clearly, whatever follows from the axioms must hold not only in E^1 but also in any other ordered field. Thus we shall henceforth state our definitions and theorems in a more general way, speaking of ordered fields in general instead of E^1 alone.

Definition 2.

An element x of an ordered field is said to be *positive* if $x > 0$ or *negative* if $x < 0$.

Here and below, "$x > y$" means the same as "$y < x$." We also write "$x \leq y$" for "$x < y$ or $x = y$"; similarly for "$x \geq y$."

Definition 3.

For any elements x, y of a field, we define their *difference*

$$x - y = x + (-y).$$

If $y \neq 0$, we also define the *quotient of x by y*

$$\frac{x}{y} = xy^{-1},$$

also denoted by x/y.

Note 4. *Division by* 0 *remains undefined.*

Definition 4.

For any element x of an ordered field, we define its *absolute value*,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \text{ and} \\ -x & \text{if } x < 0. \end{cases}$$

It follows that $|x| \geq 0$ *always*; for if $x \geq 0$, then

$$|x| = x \geq 0;$$

and if $x < 0$, then

$$|x| = -x > 0. \quad \text{(Why?)}$$

Moreover,

$$-|x| \leq x \leq |x|,$$

for,

$$\text{if } x \geq 0, \text{ then } |x| = x;$$

and

$$\text{if } x < 0, \text{ then } x < |x| \text{ since } |x| > 0.$$

Thus, in all cases,

$$x \leq |x|.$$

Similarly one shows that

$$-|x| \leq x.$$

As we have noted, all rules of arithmetic (dealing with the four arithmetic operations and inequalities) can be deduced from Axioms I through IX and thus apply to *all* ordered fields, along with E^1. We shall not dwell on their deduction, limiting ourselves to a few simple corollaries as examples.[2]

[2] For more examples, see the author's *Basic Concepts of Mathematics*, Chapter 2, §§3–4.

Corollary 1 (rule of signs).

(i) $a(-b) = (-a)b = -(ab)$;

(ii) $(-a)(-b) = ab$.

Proof. By Axiom VI,

$$a(-b) + ab = a[(-b) + b] = a \cdot 0 = 0.$$

Thus

$$a(-b) + ab = 0.$$

By definition, then, $a(-b)$ *is the additive inverse of* ab, i.e.,

$$a(-b) = -(ab).$$

Similarly, we show that

$$(-a)b = -(ab)$$

and that

$$-(-a) = a.$$

Finally, (ii) is obtained from (i) when a is replaced by $-a$. □

Corollary 2. *In an* ordered *field*, $a \neq 0$ *implies*

$$a^2 = (a \cdot a) > 0.$$

(*Hence* $1 = 1^2 > 0$.)

Proof. If $a > 0$, we may multiply by a (Axiom IX(b)) to obtain

$$a \cdot a > 0 \cdot a = 0, \text{ i.e., } a^2 > 0.$$

If $a < 0$, then $-a > 0$; so we may multiply the inequality $a < 0$ by $-a$ and obtain

$$a(-a) < 0(-a) = 0;$$

i.e., by Corollary 1,

$$-a^2 < 0,$$

whence

$$a^2 > 0. \quad \square$$

§§5–6. Natural Numbers. Induction

The element 1 was introduced in Axiom IV(b). Since addition is also assumed known, we can use it to define, step by step, the elements

$$2 = 1 + 1, \ 3 = 2 + 1, \ 4 = 3 + 1, \ \text{etc.}$$

If this process is continued indefinitely, we obtain what is called the set N of all *natural elements* in the given field F. In particular, the natural elements of E^1 are called *natural numbers*. Note that

$$(\forall n \in N) \quad n+1 \in N.$$

*A more precise approach to natural elements is as follows.[1] A subset S of a field F is said to be *inductive* iff

(i) $1 \in S$ and

(ii) $(\forall x \in S)\ x+1 \in S$.

Such subsets certainly exist; e.g., the entire field F is inductive since

$$1 \in F \text{ and } (\forall x \in F)\ x+1 \in F.$$

Define N as the intersection of *all* inductive sets in F.

***Theorem 1.** *The set N so defined is inductive itself. In fact, it is the "smallest" inductive subset of F (i.e., contained in any other such subset).*

Proof. We have to show that

(i) $1 \in N$, and

(ii) $(\forall x \in N)\ x+1 \in N$.

Now, by definition, the unity 1 is in *each* inductive set; hence it also belongs to the intersection of such sets, i.e., to N. Thus $1 \in N$, as claimed.

Next, take any $x \in N$. Then, by our definition of N, x is in *each* inductive set S; thus, by property (ii) of such sets, also $x+1$ is in each such S; hence $x+1$ is in the intersection of *all* inductive sets, i.e.,

$$x+1 \in N,$$

and so N is inductive, indeed.

Finally, by definition, N is the *common part* of all such sets and hence contained in each. □

For applications, Theorem 1 is usually expressed as follows.

Theorem 1′ (first induction law). *A proposition $P(n)$ involving a natural n holds for all $n \in N$ in a field F if*

(i) *it holds for $n = 1$, i.e., $P(1)$ is true*; *and*

(ii) *whenever $P(n)$ holds for $n = m$, it holds for $n = m+1$, i.e.,*

$$P(m) \Longrightarrow P(m+1).$$

[1] At a first reading, one may omit all "starred" passages and simply assume Theorems 1′ and 2′ below as additional axioms, without proof.

***Proof.** Let S be the set of all those $n \in N$ for which $P(n)$ is true,

$$S = \{n \in N \mid P(n)\}.$$

We have to show that actually *each* $n \in N$ is in S, i.e., $N \subseteq S$.

First, we show that S is *inductive*.

Indeed, by assumption (i), $P(1)$ is true; so $1 \in S$.

Next, let $x \in S$. This means that $P(x)$ is true. By assumption (ii), however, this implies $P(x+1)$, i.e., $x+1 \in S$. Thus

$$1 \in S \text{ and } (\forall\, x \in S)\ x+1 \in S;$$

S *is* inductive.

Then, by Theorem 1 (second clause), $N \subseteq S$, and all is proved. □

This theorem is used to prove various properties of N "*by induction.*"

Examples.

(a) *If* $m, n \in N$, *then also* $m + n \in N$ *and* $mn \in N$.

To prove the first property, fix any $m \in N$. Let $P(n)$ mean

$$m + n \in N \quad (n \in N).$$

Then

(i) $P(1)$ *is true*, for as $m \in N$, the definition of N yields $m + 1 \in N$, i.e., $P(1)$.

(ii) $P(k) \Rightarrow P(k+1)$ *for* $k \in N$. Indeed,

$$\begin{aligned} P(k) &\Rightarrow m + k \in N \Rightarrow (m+k) + 1 \in N \\ &\Rightarrow m + (k+1) \in N \Rightarrow P(k+1). \end{aligned}$$

Thus, by Theorem 1′, $P(n)$ holds for *all* n; i.e.,

$$(\forall\, n \in N) \quad m + n \in N$$

for any $m \in N$.

To prove the same for mn, we let $P(n)$ mean

$$mn \in N \quad (n \in N)$$

and proceed similarly.

(b) *If* $n \in N$, *then* $n - 1 = 0$ *or* $n - 1 \in N$.

For an inductive proof, let $P(n)$ mean

$$n - 1 = 0 \text{ or } n - 1 \in N \quad (n \in N).$$

Then proceed as in (a).

(c) *In an ordered field, all naturals are* ≥ 1.

Indeed, let $P(n)$ mean that

$$n \geq 1 \quad (n \in N).$$

Then

(i) $P(1)$ *holds* since $1 = 1$.

(ii) $P(m) \Rightarrow P(m+1)$ for $m \in N$, since

$$P(m) \Rightarrow m \geq 1 \Rightarrow (m+1) > 1 \Rightarrow P(m+1).$$

Thus Theorem 1′ yields the result.

(d) *In an ordered field,* $m, n \in N$ *and* $m > n$ *implies* $m - n \in N$.

For an inductive proof, fix any $m \in N$ and let $P(n)$ mean

$$m - n \leq 0 \text{ or } m - n \in N \quad (n \in N).$$

Use (b).

(e) *In an ordered field,* $m, n \in N$ *and* $m < n + 1$ *implies* $m \leq n$.

For, by (d), $m > n$ would imply $m - n \in N$, hence $m - n \geq 1$, or $m \geq n + 1$, contrary to $m < n + 1$.

Our next theorem states the so-called *well-ordering property of* N.

Theorem 2 (well-ordering of N)**.** *In an ordered field, each nonvoid set* $A \subseteq N$ *has a least member (i.e., one that exceeds no other element of* A*).*

Proof outline.[2] Given $\emptyset \neq A \subseteq N$, let $P(n)$ be the proposition "*Any subset of* A *containing elements* $\leq n$ *has a least member*" $(n \in N)$. Use Theorem 1′ and Example (e). □

This theorem yields a new form of the induction law.

Theorem 2′ (second induction law)**.** *A proposition* $P(n)$ *holds for all* $n \in N$ *in an ordered field if*

(i′) $P(1)$ *holds and*

(ii′) *whenever* $P(n)$ *holds for all naturals less than some* $m \in N$, *then* $P(n)$ *also holds for* $n = m$.

Proof. Assume (i′) and (ii′). Seeking a contradiction,[3] suppose there are some $n \in N$ (call them "bad") for which $P(n)$ *fails*. Then these "bad" naturals form a *nonvoid* subset of N, call it A.

[2] For a more detailed proof, see *Basic Concepts of Mathematics*, Chapter 2, §5, Theorem 2.

[3] We are using a "proof by contradiction" or "indirect proof." Instead of proving our assertion *directly*, we show that the *opposite is impossible*, being contradictory.

By Theorem 2, A has a *least* member m. Thus m is the *least* natural for which $P(n)$ fails. It follows that all n *less* than m do satisfy $P(n)$. But then, by our assumption (ii′), $P(n)$ also holds for $n = m$, which is impossible for, by construction, m is "bad" (it is in A). This contradiction shows that there are no "bad" naturals. Thus all is proved. □

Note 1. All the preceding arguments hold also if, in our definition of N and all formulations, the unity 1 is replaced by 0 or by some k ($\pm k \in N$). Then, however, the conclusions must be changed to say that $P(n)$ holds for all integers $n \geq k$ (instead of "$n \geq 1$"). We then say that "induction *starts* with k."

An analogous induction law also applies to *definitions of concepts* $C(n)$.

A notion $C(n)$ *involving a natural* n *is regarded as defined for each* $n \in N$ (*in* E^1) *if*

(i) *it is defined for* $n = 1$ *and*

(ii) *some rule is given that expresses* $C(n+1)$ *in terms of* $C(1), \ldots, C(n)$.

(Note 1 applies here, too.)

$C(n)$ itself need not be a *number*; it may be of quite general nature.

We shall adopt this principle as a kind of logical axiom, without proof (though it can be proved in a similar manner as Theorems 1′ and 2′). The underlying intuitive idea is a "step-by-step" process—first, we define $C(1)$; then, as $C(1)$ is known, we may use it to define $C(2)$; next, once both are known, we may use them to define $C(3)$; and so on, ad infinitum. Definitions based on that principle are called *inductive* or *recursive*. The following examples are important.

Examples (continued)**.**

(f) For any element x of a field, we define its *nth power* x^n and its *n-multiple* nx by

(i) $x^1 = 1x = x$;

(ii) $x^{n+1} = x^n x$ (respectively, $(n+1)x = nx + x$).

We may think of it as a step-by-step definition:

$$x^1 = x,\ x^2 = x^1 x,\ x^3 = x^2 x,\ \text{etc.}$$

(g) For each natural number n, we define its *factorial* $n!$ by

$$1! = 1,\ (n+1)! = n!\,(n+1);$$

e.g., $2! = 1!\,(2) = 2$, $3! = 2!\,(3) = 6$, etc. We also define $0! = 1$.

(h) The *sum* and *product of n field elements* $x_1, x_2, \ldots, x_n$, denoted by

$$\sum_{k=1}^{n} x_k \text{ and } \prod_{k=1}^{n} x_k$$

or

$$x_1 + x_2 + \cdots + x_n \text{ and } x_1 x_2 \cdots x_n, \text{ respectively,}$$

are defined recursively.

Sums are defined by

(i) $\displaystyle\sum_{k=1}^{1} x_k = x_1;$

(ii) $\displaystyle\sum_{k=1}^{n+1} x_k = \left(\sum_{k=1}^{n} x_k\right) + x_{n+1},\ n = 1, 2, \ldots.$

Thus

$$x_1 + x_2 + x_3 = (x_1 + x_2) + x_3,$$
$$x_1 + x_2 + x_3 + x_4 = (x_1 + x_2 + x_3) + x_4, \text{ etc.}$$

Products are defined by

(i) $\displaystyle\prod_{k=1}^{1} x_k = x_1;$

(ii) $\displaystyle\prod_{k=1}^{n+1} x_k = \left(\prod_{k=1}^{n} x_k\right) \cdot x_{n+1}.$

(i) Given any objects $x_1, x_2, \ldots, x_n, \ldots$, the *ordered n-tuple*

$$(x_1, x_2, \ldots, x_n)$$

is defined inductively by

(i) $(x_1) = x_1$ (i.e., the ordered "one-tuple" (x_1) is x_1 itself) and

(ii) $(x_1, x_2, \ldots, x_{n+1}) = ((x_1, \ldots, x_n), x_{n+1})$, i.e., the ordered $(n+1)$-tuple is a *pair* (y, x_{n+1}) in which the first term y is itself an ordered n-tuple, $(x_1, \ldots, x_n)$; for example,

$$(x_1, x_2, x_3) = ((x_1, x_2), x_3), \text{ etc.}$$

Problems on Natural Numbers and Induction

1. Complete the missing details in Examples (a), (b), and (d).

2. Prove Theorem 2 in detail.

3. Suppose $x_k < y_k$, $k = 1, 2, \ldots$, in an ordered field. Prove by induction on n that

(a) $\sum_{k=1}^{n} x_k < \sum_{k=1}^{n} y_k$;

(b) if all x_k, y_k are greater than zero, then

$$\prod_{k=1}^{n} x_k < \prod_{k=1}^{n} y_k.$$

4. Prove by induction that

(i) $1^n = 1$;

(ii) $a < b \Rightarrow a^n < b^n$ if $a > 0$.

Hence deduce that

(iii) $0 \le a^n < 1$ if $0 \le a < 1$;

(iv) $a^n < b^n \Rightarrow a < b$ if $b > 0$; proof by contradiction.

5. Prove the *Bernoulli inequalities*: For any element ε of an ordered field,

(i) $(1+\varepsilon)^n \ge 1 + n\varepsilon$ if $\varepsilon > -1$;

(ii) $(1-\varepsilon)^n \ge 1 - n\varepsilon$ if $\varepsilon < 1$; $n = 1, 2, 3, \ldots$.

6. For any field elements a, b and natural *numbers* m, n, prove that

(i) $a^m a^n = a^{m+n}$; (ii) $(a^m)^n = a^{mn}$;

(iii) $(ab)^n = a^n b^n$; (iv) $(m+n)a = ma + na$;

(v) $n(ma) = (nm) \cdot a$; (vi) $n(a+b) = na + nb$.

[Hint: For problems involving two natural numbers, fix m and use induction on n].

7. Prove that in any field,

$$a^{n+1} - b^{n+1} = (a-b) \sum_{k=0}^{n} a^k b^{n-k}, \quad n = 1, 2, 3, \ldots.$$

Hence for $r \ne 1$

$$\sum_{k=0}^{n} ar^k = a\,\frac{1 - r^{n+1}}{1-r}$$

(sum of n terms of a geometric series).

8. For $n > 0$ define

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!\,(n-k)!}, & 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Verify *Pascal's law*,

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

Then prove by induction on n that

(i) $(\forall\, k \mid 0 \le k \le n)\ \binom{n}{k} \in N$; and

(ii) for any field elements a and b,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \quad n \in N \text{ (the \textit{binomial theorem})}.$$

What value must 0^0 take for (ii) to hold for all a and b?

9. Show by induction that in an ordered field F any *finite* sequence $x_1, \ldots, x_n$ has a largest and a least term (which *need not* be x_1 or x_n). Deduce that *all* of N is an *infinite* set, in any ordered field.

10. Prove in E^1 that

(i) $\displaystyle\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$;

(ii) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$;

(iii) $\displaystyle\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$;

(iv) $\displaystyle\sum_{k=1}^{n} k^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$.

§7. Integers and Rationals

All natural elements of a field F, their additive inverses, and 0 are called the *integral elements* of F, briefly *integers*.

An element $x \in F$ is said to be *rational* iff $x = \dfrac{p}{q}$ for some *integers* p and q $(q \neq 0)$; x is *irrational* iff it is not rational.

We denote by J the set of all integers, and by R the set of all rationals, in F. *Every integer p is also a rational* since p can be written as p/q with $q = 1$. Thus

$$R \supseteq J \supset N.$$

In an *ordered* field,

$$N = \{x \in J \mid x > 0\}. \text{ (Why?)}$$

Theorem 1. *If a and b are integers (or rationals) in F, so are $a + b$ and ab.*

Proof. For integers, this follows from Examples (a) and (d) in §§5–6; one only has to distinguish three cases:

(i) $a, b \in N$;

(ii) $-a \in N$, $b \in N$;

(iii) $a \in N$, $-b \in N$.

The details are left to the reader (see *Basic Concepts of Mathematics*, Chapter 2, §7, Theorem 1).

Now let a and b be rationals, say,

$$a = \frac{p}{q} \text{ and } b = \frac{r}{s},$$

where $p, q, r, s \in J$ and $q, s \neq 0$. Then, as is easily seen,

$$a \pm b = \frac{ps \pm qr}{qs} \text{ and } ab = \frac{pr}{qs},$$

where $qs \neq 0$; and qs and pr are *integers* by the first part of the proof (since $p, q, r, s \in J$).

Thus $a \pm b$ and ab are fractions with integral numerators and denominators. Hence, by definition, $a \pm b \in R$ and $ab \in R$. □

Theorem 2. *In any field F, the set R of all rationals is a field itself, under the operations defined in F, with the same neutral elements 0 and 1. Moreover, R is an* ordered *field if F is. (We call R the* rational subfield *of F.)*

Proof. We have to check that R satisfies the field axioms.

The closure law I follows from Theorem 1.

Axioms II, III, and VI hold for rationals because they hold for *all* elements of F; similarly for Axioms VII to IX if F is ordered.

Axiom IV holds in R because the neutral elements 0 and 1 *belong to* R; indeed, they are integers, hence certainly rationals.

To verify Axiom V, we must show that $-x$ and x^{-1} *belong to* R if x does. If, however,

$$x = \frac{p}{q} \quad (p, q \in J,\ q \neq 0),$$

then

$$-x = \frac{-p}{q},$$

where again $-p \in J$ by the definition of J; thus $-x \in R$.

If, in addition, $x \neq 0$, then $p \neq 0$, and

$$x = \frac{p}{q} \text{ implies } x^{-1} = \frac{q}{p}. \text{ (Why?)}$$

Thus $x^{-1} \in R$. $\square$

Note. The representation

$$x = \frac{p}{q} \quad (p,\, q \in J)$$

is not unique in general; in an *ordered* field, however, we can always choose $q > 0$, i.e., $q \in N$ (take $p \leq 0$ if $x \leq 0$).

Among all such q there is a *least* one by Theorem 2 of §§5–6. If $x = p/q$, with this *minimal* $q \in N$, we say that the rational x is given in *lowest terms*.

§§8–9. Upper and Lower Bounds. Completeness Axiom

A subset A of an ordered field F is said to be *bounded below* (or *left bounded*) iff there is $p \in F$ such that

$$(\forall\, x \in A) \quad p \leq x;$$

A is *bounded above* (or *right bounded*) iff there is $q \in F$ such that

$$(\forall\, x \in A) \quad x \leq q.$$

In this case, p and q are called, respectively, a *lower* (or *left*) bound and an *upper* (or *right*) bound, of A. If *both* exist, we simply say that A is *bounded* (by p and q). The empty set $\emptyset$ is regarded as ("vacuously") bounded by *any* p and q (cf. the end of Chapter 1, §3).

The bounds p and q may, but *need not*, belong to A. If a left bound p is itself in A, we call it the *least element* or *minimum* of A, denoted $\min A$. Similarly, if A *contains* an upper bound q, we write $q = \max A$ and call q the *largest element* or *maximum* of A. However, A may well have no minimum or maximum.

Note 1. A *finite* set $A \neq \emptyset$ always has a minimum and a maximum (see Problem 9 of §§5–6).

Note 2. A set A can have *at most one* maximum and at most one minimum. For if it had *two* maxima q, q', then

$$q \leq q'$$

(since $q \in A$ and q' is a right bound); similarly

$$q' \leq q;$$

so $q = q'$ after all. Uniqueness of $\min A$ is proved in the same manner.

Note 3. If A has *one* lower bound p, it has *many* (e.g., take any $p' < p$). Similarly, if A has *one* upper bound q, it has *many* (take any $q' > q$). Geometrically, on the real axis, all lower (upper) bounds lie to the left (right) of A; see Figure 1.

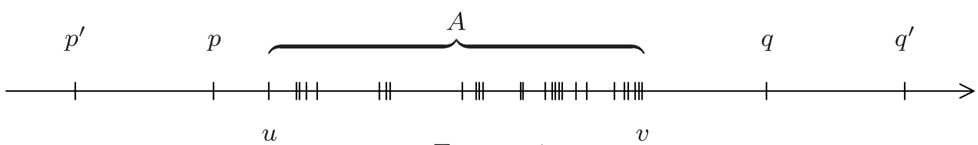

FIGURE 1

Examples.

(1) Let

$$A = \{1, -2, 7\}.$$

Then A is bounded above (e.g., by $7, 8, 10, \ldots$) and below (e.g., by $-2, -5, -12, \ldots$).

We have $\min A = -2$, $\max A = 7$.

(2) The set N of all naturals is bounded below (e.g., by $1, 0, \frac{1}{2}, -1, \ldots$), and $1 = \min N$; N has no maximum, for each $q \in N$ is *exceeded* by some $n \in N$ (e.g., $n = q + 1$).

(3) Given $a, b \in F$ $(a \leq b)$, we define in F the *open interval*

$$(a, b) = \{x \mid a < x < b\};$$

the *closed interval*

$$[a, b] = \{x \mid a \leq x \leq b\};$$

the *half-open interval*

$$(a, b] = \{x \mid a < x \leq b\};$$

and the *half-closed interval*

$$[a, b) = \{x \mid a \leq x < b\}.$$

Clearly, each of these intervals is bounded by the *endpoints* a and b; moreover, $a \in [a, b]$ and $a \in [a, b)$ (the latter provided $[a, b) \neq \emptyset$, i.e., $a < b$), and $a = \min[a, b] = \min[a, b)$; similarly, $b = \max[a, b] = \max(a, b]$. But $[a, b)$ has no maximum, $(a, b]$ has no minimum, and (a, b) has neither. (Why?)

Geometrically, it seems plausible that among all left and right bounds of A (if any) there are some " closest" to A, such as u and v in Figure 1, i.e., a *least* *upper bound* v and a *greatest lower bound* u. These are abbreviated

$$\operatorname{lub} A \text{ and } \operatorname{glb} A$$

and are also called the *supremum* and *infimum* of A, respectively; briefly,

$$v = \sup A,\ u = \inf A.$$

However, this assertion, though valid in E^1, fails to materialize in many other fields such as the field R of all rationals (cf. §§11–12). Even for E^1, it cannot be *proved* from Axioms I through IX.

On the other hand, this property is of utmost importance for mathematical analysis; so we introduce it as an *axiom* (for E^1), called the *completeness axiom*. It is convenient first to give a general definition.

Definition 1.

An ordered field F is said to be *complete* iff every nonvoid right-bounded subset $A \subset F$ has a supremum (i.e., a lub) in F.

Note that we use the term "complete" only for *ordered* fields.

With this definition, we can give the tenth and final axiom for E^1.

X (completeness axiom). *The real field E^1 is complete in the above sense. That is, each right-bounded set $A \subset E^1$ has a supremum* ($\sup A$) *in E^1, provided $A \neq \emptyset$.*

The corresponding assertion for *infima* can now be *proved* as a theorem.

Theorem 1. *In a complete field F (such as E^1), every nonvoid left-bounded subset $A \subset F$ has an infimum (i.e., a glb).*

Proof. Let B be the (nonvoid) set of all lower bounds of A (such bounds *exist* since A is left bounded). Then, clearly, no member of B exceeds any member of A, and so B *is right bounded* by an element of A. Hence, by the assumed completeness of F, B *has a supremum in* F, call it p.

We shall show that p is also the required infimum of A, thus completing the proof.

Indeed, we have

(i) *p is a lower bound of A.* For, by definition, p is the *least* upper bound of B. But, as shown above, each $x \in A$ is an upper bound of B. Thus

$$(\forall x \in A) \quad p \leq x.$$

(ii) *p is the* greatest *lower bound of A.* For $p = \sup B$ is not exceeded by any member of B. But, by definition, B contains *all* lower bounds of A; so p is not exceeded by any of them, i.e.,

$$p = \operatorname{glb} A = \inf A. \quad \square$$

Note 4. The lub and glb of A (if they exist) are *unique*. For $\inf A$ is, by definition, the maximum of the set B of all lower bounds of A, and hence unique, by Note 2; similarly for the uniqueness of $\sup A$.

Note 5. Unlike $\min A$ and $\max A$, the glb and lub of A *need not* belong to A. For example, if A is the interval (a, b) in E^1 $(a < b)$ then, as is easily seen,

$$a = \inf A \text{ and } b = \sup A,$$

though $a, b \notin A$. *Thus* $\sup A$ *and* $\inf A$ *may exist*, *though* $\max A$ *and* $\min A$ *do not*.

On the other hand, if

$$q = \max A \ (p = \min A),$$

then also

$$q = \sup A \ (p = \inf A). \quad \text{(Why?)}$$

Theorem 2. *In an ordered field* F, *we have* $q = \sup A$ $(A \subset F)$ *iff*

(i) $(\forall x \in A) \quad x \leq q$ *and*

(ii) *each field element* $p < q$ *is exceeded by some* $x \in A$; *i.e.*,

$$(\forall p < q)\ (\exists x \in A) \quad p < x.$$

Equivalently,

(ii′) $$(\forall \varepsilon > 0)\ (\exists x \in A) \quad q - \varepsilon < x; \quad (\varepsilon \in F).$$

Similarly, $p = \inf A$ *iff*

$$(\forall x \in A) \quad p \leq x \quad \textit{and} \quad (\forall \varepsilon > 0)\ (\exists x \in A) \quad p + \varepsilon > x.$$

Proof. Condition (i) states that q is an upper bound of A, while (ii) implies that no *smaller* element p is such a bound (since it is *exceeded* by some x in A). When combined, (i) and (ii) state that q is the *least* upper bound.

Moreover, any element $p < q$ can be written as $q - \varepsilon$ $(\varepsilon > 0)$. Hence (ii) can be rephrased as (ii′).

The proof for $\inf A$ is quite analogous. □

Corollary 1. *Let* $b \in F$ *and* $A \subset F$ *in an ordered field* F. *If each element* x *of* A *satisfies* $x \leq b$ $(x \geq b)$, *so does* $\sup A$ $(\inf A$, *respectively*), *provided it exists in* F.

In fact, the condition

$$(\forall x \in A) \quad x \leq b$$

means that b is a right bound of A. However, $\sup A$ is the *least* right bound, so $\sup A \leq b$; similarly for $\inf A$.

Corollary 2. *In any ordered field,* $\emptyset \neq A \subseteq B$ *implies*

$$\sup A \leq \sup B \text{ and } \inf A \geq \inf B,$$

as well as

$$\inf A \leq \sup A,$$

provided the suprema and infima involved exist.

Proof. Let $p = \inf B$ and $q = \sup B$.

As q is a right bound of B,

$$x \leq q \text{ for all } x \in B.$$

But $A \subseteq B$, so B contains all elements of A. Thus

$$x \in A \Rightarrow x \in B \Rightarrow x \leq q;$$

so, by Corollary 1, also

$$\sup A \leq q = \sup B,$$

as claimed.

Similarly, one gets $\inf A \geq \inf B$.

Finally, if $A \neq \emptyset$, we can fix some $x \in A$. Then

$$\inf A \leq x \leq \sup A,$$

and all is proved. □

Problems on Upper and Lower Bounds

1. Complete the proofs of Theorem 2 and Corollaries 1 and 2 for *infima*. Prove the last clause of Note 4.

2. Prove that F is complete iff each nonvoid left-bounded set in F has an infimum.

3. Prove that if $A_1, A_2, \ldots, A_n$ are right bounded (left bounded) in F, so is

$$\bigcup_{k=1}^{n} A_k.$$

4. Prove that if $A = (a, b)$ is an open interval ($a < b$), then

$$a = \inf A \text{ and } b = \sup A.$$

5. In an ordered field F, let $\emptyset \neq A \subset F$. Let $c \in F$ and let cA denote the set of all products cx ($x \in A$); i.e.,

$$cA = \{cx \mid x \in A\}.$$

Prove that

(i) if $c \geq 0$, then

$$\sup(cA) = c \cdot \sup A \text{ and } \inf(cA) = c \cdot \inf A;$$

(ii) if $c < 0$, then

$$\sup(cA) = c \cdot \inf A \text{ and } \inf(cA) = c \cdot \sup A.$$

In both cases, assume that the right-side $\sup A$ (respectively, $\inf A$) exists.

6. From Problem 5(ii) with $c = -1$, obtain a new proof of Theorem 1.
[Hint: If A is left bounded, show that $(-1)A$ is right bounded and use its supremum.]

7. Let A and B be subsets of an ordered field F. Assuming that the required lub and glb exist in F, prove that

(i) if $(\forall x \in A)\ (\forall y \in B)\ x \leq y$, then $\sup A \leq \inf B$;

(ii) if $(\forall x \in A)\ (\exists y \in B)\ x \leq y$, then $\sup A \leq \sup B$;

(iii) if $(\forall y \in B)\ (\exists x \in A)\ x \leq y$, then $\inf A \leq \inf B$.

[Hint for (i): By Corollary 1, $(\forall y \in B)\ \sup A \leq y$, so $\sup A \leq \inf B$. (Why?)]

8. For any two subsets A and B of an ordered field F, let $A + B$ denote the set of all sums $x + y$ with $x \in A$ and $y \in B$; i.e.,

$$A + B = \{x + y \mid x \in A,\ y \in B\}.$$

Prove that if $\sup A = p$ and $\sup B = q$ exist in F, then

$$p + q = \sup(A + B);$$

similarly for infima.
[Hint for sup: By Theorem 2, we must show that

(i) $(\forall x \in A)\ (\forall y \in B)\ x + y \leq p + q$ (which is easy) and

(ii′) $(\forall \varepsilon > 0)\ (\exists\ x \in A)\ (\exists\ y \in B)\ x + y > (p + q) - \varepsilon$.

Fix any $\varepsilon > 0$. By Theorem 2,

$$(\exists\ x \in A)\ (\exists\ y \in B) \quad p - \frac{\varepsilon}{2} < x \text{ and } q - \frac{\varepsilon}{2} < y. \text{ (Why?)}$$

Then

$$x + y > \left(p - \frac{\varepsilon}{2}\right) + \left(q - \frac{\varepsilon}{2}\right) = (p + q) - \varepsilon,$$

as required.]

9. In Problem 8 let A and B consist of *positive* elements only, and let

$$AB = \{xy \mid x \in A,\ y \in B\}.$$

Prove that if $\sup A = p$ and $\sup B = q$ exist in F, then

$$pq = \sup(AB);$$

similarly for infima.

[Hint: Use again Theorem 2(ii′). For sup(AB), take

$$0 < \varepsilon < (p+q)\min\{p,\, q\}$$

and

$$x > p - \frac{\varepsilon}{p+q} \text{ and } y > q - \frac{\varepsilon}{p+q};$$

show that

$$xy > pq - \varepsilon + \frac{\varepsilon^2}{(p+q)^2} > pq - \varepsilon.$$

For $\inf(AB)$, let $s = \inf B$ and $r = \inf A$; choose $d < 1$, with

$$0 < d < \frac{\varepsilon}{1+r+s}.$$

Now take $x \in A$ and $y \in B$ with

$$x < r + d \text{ and } y < s + d,$$

and show that

$$xy < rs + \varepsilon.$$

Explain!]

10. Prove that

(i) if $(\forall\, \varepsilon > 0)\ a \geq b - \varepsilon$, then $a \geq b$;

(ii) if $(\forall\, \varepsilon > 0)\ a \leq b + \varepsilon$, then $a \leq b$.

11. Prove the *principle of nested intervals*: If $[a_n,\, b_n]$ are closed intervals in a *complete* ordered field F, with

$$[a_n,\, b_n] \supseteq [a_{n+1},\, b_{n+1}], \quad n = 1, 2, \ldots,$$

then

$$\bigcap_{n=1}^{\infty} [a_n,\, b_n] \neq \emptyset.$$

[Hint: Let

$$A = \{a_1,\, a_2,\, \ldots,\, a_n,\, \ldots\}.$$

Show that A is bounded above by each b_n.

Let $p = \sup A$. (Does it exist?)

Show that

$$(\forall\, n) \quad a_n \leq p \leq b_n,$$

i.e.,

$$p \in [a_n,\, b_n].]$$

12. Prove that each bounded set $A \neq \emptyset$ in a complete field F is contained in a *smallest* closed interval $[a, b]$ (so $[a, b]$ is contained in any other $[c, d] \supseteq A$).

Show that this fails if "closed" is replaced by "open."

[Hint: Take $a = \inf A$, $b = \sup A$].

13. Prove that if A consists of positive elements only, then $q = \sup A$ iff

(i) $(\forall\, x \in A)\ x \leq q$ and

(ii) $(\forall\, d > 1)\ (\exists\, x \in A)\ q/d < x$.

[Hint: Use Theorem 2.]

§10. Some Consequences of the Completeness Axiom

The ancient Greek geometer and scientist Archimedes was first to observe that even a large distance y can be measured by a small yardstick x; one only has to mark x off sufficiently many times. Mathematically, this means that, given any $x > 0$ and any y, there is an $n \in N$ such that $nx > y$. This fact, known as the *Archimedean property*, holds not only in E^1 but also in many other ordered fields. Such fields are called *Archimedean*. In particular, we have the following theorem.

Theorem 1. *Any complete field F (e.g., E^1) is Archimedean.*[1]

That is, given any $x, y \in F$ $(x > 0)$ in such a field, there is a natural $n \in F$ such that $nx > y$.

Proof by contradiction. Suppose this fails. Thus, given $y, x \in F$ $(x > 0)$, assume that there is *no* $n \in N$ with $nx > y$.

Then

$$(\forall\, n \in N) \quad nx \leq y;$$

i.e., y is an upper bound of the set of *all* products nx $(n \in N)$. Let

$$A = \{nx \mid n \in N\}.$$

Clearly, A is bounded above (by y) and $A \neq \emptyset$; so, by the assumed completeness of F, A has a supremum, say, $q = \sup A$.

As q is an upper bound, we have (by the definition of A) that $nx \leq q$ for *all* $n \in N$, hence also $(n+1)x \leq q$; i.e.,

$$nx \leq q - x$$

for all $n \in N$ (since $n \in N \Rightarrow n + 1 \in N$).

[1] However, there also are *incomplete* Archimedean fields (see Note 2 in §§11–12).

Thus $q - x$ (which is *less* than q for $x > 0$) *is another upper bound of all* nx, i.e., of the set A.

This is impossible, however, since $q = \sup A$ is the *least* upper bound of A.

This contradiction completes the proof. □

Corollary 1. *In any Archimedean (hence also in any complete) field* F, *the set* N *of all natural elements has no upper bounds, and the set* J *of all integers has neither upper nor lower bounds. Thus*

$$(\forall y \in F)\ (\exists\, m, n \in N) \quad -m < y < n.$$

Proof. Given any $y \in F$, one can use the Archimedean property (with $x = 1$) to find an $n \in N$ such that

$$n \cdot 1 > y, \text{ i.e., } n > y.$$

Similarly, there is an $m \in N$ such that

$$m > -y, \text{ i.e., } -m < y.$$

This proves our *last* assertion and shows that *no* $y \in F$ can be a right bound of N (for $y < n \in N$), or a left bound of J (for $y > -m \in J$). □

Theorem 2. *In any Archimedean (hence also in any complete) field* F, *each left (right) bounded set* A *of* integers $(\emptyset \neq A \subset J)$ *has a minimum (maximum, respectively).*

Proof. Suppose $\emptyset \neq A \subseteq J$, and A has a *lower* bound y.

Then Corollary 1 (last part) yields a *natural* m, with $-m < y$, so that

$$(\forall x \in A) \quad -m < x,$$

and so $x + m > 0$.

Thus, by adding m to each $x \in A$, we obtain a set (call it $A+m$) of *naturals*.[2]

Now, by Theorem 2 of §§5–6, $A + m$ has a minimum; call it p. As p is the least of all sums $x + m$, $p - m$ is the least of all $x \in A$; so $p - m = \min A$ exists, as claimed.

Next, let A have a *right* bound z. Then look at the set of all *additive inverses* $-x$ of points $x \in A$; call it B.

Clearly, B is left bounded (by $-z$), so it has a minimum, say, $u = \min B$. Then $-u = \max A$. (Verify!) □

In particular, given any $x \in F$ (F Archimedean), let $[x]$ denote the greatest integer $\leq x$ (called the *integral part* of x). We thus obtain the following corollary.

[2] This is the main point—geometrically, we have "shifted" A to the right by m, so that its elements became *positive* integers: $A + m \subseteq N$.

Corollary 2. *Any element x of an Archimedean field F has an integral part $[x]$. It is the unique integer n such that*

$$n \leq x < n+1.$$

(*It exists*, *by Theorem* 2.)

Any ordered field has the so-called *density property*:

If $a < b$ in F, there is $x \in F$ such that $a < x < b$; e.g., take

$$x = \frac{a+b}{2}.$$

We shall now show that, in *Archimedean* fields, x can be chosen *rational*, even if a and b are not. We refer to this as the *density of rationals* in an Archimedean field.

Theorem 3 (density of rationals)**.** *Between any elements a and b $(a < b)$ of an Archimedean field F (such as E^1), there is a rational $r \in F$ with*

$$a < r < b.$$

Proof. Let $p = [a]$ (the integral part of a). The idea of the proof is to start with p and to mark off a small "yardstick"

$$\frac{1}{n} < b - a$$

several (m) times, until

$$p + \frac{m}{n} \text{ lands inside } (a, b);$$

then $r = p + \frac{m}{n}$ is the desired rational.

We now make it precise. As F is Archimedean, there are $m, n \in N$ such that

$$n(b-a) > 1 \text{ and } m\left(\frac{1}{n}\right) > a - p.$$

We fix the *least* such m (it exists, by Theorem 2 in §§5–6). Then

$$a - p < \frac{m}{n}, \text{ but } \frac{m-1}{n} \leq a - p$$

(by the *minimality* of m). Hence

$$a < p + \frac{m}{n} \leq a + \frac{1}{n} < a + (b-a),$$

since $\frac{1}{n} < b - a$. Setting

$$r = p + \frac{m}{n},$$

we find

$$a < r < a + b - a = b. \quad \square$$

Note. Having found one rational r_1,

$$a < r_1 < b,$$

we can apply Theorem 3 to find another $r_2 \in R$,

$$r_1 < r_2 < b,$$

then a third $r_3 \in R$,

$$r_2 < r_3 < b,$$

and so on. Continuing this process indefinitely, we obtain *infinitely many* rationals in (a, b).

§§11–12. Powers With Arbitrary Real Exponents. Irrationals

In *complete* fields, one can define a^r for any $a > 0$ and $r \in E^1$ (for $r \in N$, see §§5–6, Example (f)). First of all, we have the following theorem.

Theorem 1. *Given $a \geq 0$ in a complete field F, and a natural number $n \in E^1$, there always is a unique element $p \in F$, $p \geq 0$, such that*

$$p^n = a.$$

It is called the nth root of a, *denoted*

$$\sqrt[n]{a} \text{ or } a^{1/n}.$$

(*Note that $\sqrt[n]{a} \geq 0$, by definition.*)

A direct proof, from the completeness axiom, is sketched in Problems 1 and 2 below. We shall give a simpler proof in Chapter 4, §9, Example (a). At present, we omit it and temporarily take Theorem 1 for granted. Hence we obtain the following result.

Theorem 2. *Every complete field F (such as E^1) has irrational elements, i.e., elements that are not rational.*

In particular, $\sqrt{2}$ is irrational.[1]

Proof. By Theorem 1, F *has* the element

$$p = \sqrt{2} \text{ with } p^2 = 2.$$

[1] As usual, we write $\sqrt{a}$ for $\sqrt[2]{a}$.

Seeking a contradiction, suppose $\sqrt{2}$ is rational, i.e.,

$$\sqrt{2} = \frac{m}{n}$$

for some $m, n \in N$ in *lowest terms* (see §7, final note).

Then m and n are not *both* even (otherwise, reduction by 2 would yield a *smaller* n). From $m/n = \sqrt{2}$, we obtain

$$m^2 = 2n^2;$$

so m^2 is *even*.

Only even elements have *even* squares, however.[2] Thus m itself must be even; i.e., $m = 2r$ for some $r \in N$. It follows that

$$4r^2 = m^2 = 2n^2, \text{ i.e., } 2r^2 = n^2$$

and, by the same argument, *n must be even.*

This contradicts the fact that m and n are not *both* even, and this contradiction shows that $\sqrt{2}$ must be irrational. □

Note 1. Similarly, one can prove the irrationality of $\sqrt{a}$ where $a \in N$ and a is not the square of a natural. See Problem 3 below for a hint.

Note 2. Theorem 2 shows that *the field R of all rationals is not complete* (for it contains no irrationals), even though it is *Archimedean* (see Problem 6). Thus the *Archimedean property does not imply completeness* (but see Theorem 1 of §10).

Next, we define a^r for any *rational* number $r > 0$.

Definition 1.

Given $a \geq 0$ in a complete field F, and a rational number

$$r = \frac{m}{n} \quad (m, n \in N \subseteq E^1),$$

we define

$$a^r = \sqrt[n]{a^m}.$$

Here we must clarify two facts.

(1) If $n = 1$, we have

$$a^r = a^{m/1} = \sqrt[1]{a^m} = a^m.$$

[2] For if m is *odd*, then $m = 2q - 1$ for some $q \in N$, and hence

$$m^2 = (2q-1)^2 = 4q^2 - 4q + 1 = 4q(q-1) + 1$$

is an *odd* number.

If $m = 1$, we get

$$a^r = a^{1/n} = \sqrt[n]{a}.$$

Thus Definition 1 agrees with our previous definitions of a^m and $\sqrt[n]{a}$ $(m, n \in N)$.

(2) If r is written as a fraction *in two different ways*,

$$r = \frac{m}{n} = \frac{p}{q},$$

then, as is easily seen,

$$\sqrt[n]{a^m} = \sqrt[q]{a^p} = a^r,$$

and so *our definition is unambiguous* (independent of the particular representation of r).

Indeed,

$$\frac{m}{n} = \frac{p}{q} \text{ implies } mq = np,$$

whence

$$a^{mq} = a^{pn},$$

i.e.,

$$(a^m)^q = (a^p)^n;$$

cf. §§5–6, Problem 6.

By definition, however,

$$(\sqrt[n]{a^m})^n = a^m \text{ and } (\sqrt[q]{a^p})^q = a^p.$$

Substituting this in $(a^m)^q = (a^p)^n$, we get

$$(\sqrt[n]{a^m})^{nq} = (\sqrt[q]{a^p})^{nq},$$

whence

$$\sqrt[n]{a^m} = \sqrt[q]{a^p}.$$

Thus Definition 1 is valid, indeed.

By using the results of Problems 4 and 6 of §§5–6, the reader will easily obtain analogous formulas for powers with positive *rational* exponents, namely,

$$\begin{gathered} a^r a^s = a^{r+s};\ (a^r)^s = a^{rs};\ (ab)^r = a^r b^r;\ a^r < a^s \text{ if } 0 < a < 1 \text{ and } r > s; \\ a < b \text{ iff } a^r < b^r\ (a,\, b,\, r > 0);\ a^r > a^s \text{ if } a > 1 \text{ and } r > s;\ 1^r = 1 \end{gathered} \tag{1}$$

Henceforth we assume these formulas known, for *rational* $r, s > 0$.

Next, we define a^r for any *real* $r > 0$ and any element $a > 1$ in a complete field F.

Let A_{ar} denote the set of all members of F of the form a^x, with $x \in R$ and $0 < x \leq r$; i.e.,

$$A_{ar} = \{a^x \mid 0 < x \leq r,\ x \text{ rational}\}.$$

By the density of rationals in E^1 (Theorem 3 of §10), such rationals x do exist; thus $A_{ar} \neq \emptyset$.

Moreover, A_{ar} is *right bounded* in F. Indeed, fix any rational number $y > r$. By the formulas in (1), we have, for any positive *rational* $x \leq r$,

$$a^y = a^{x+(y-x)} = a^x a^{y-x} > a^x$$

since $a > 1$ and $y - x > 0$ implies

$$a^{y-x} > 1.$$

Thus a^y *is an upper bound of all* a^x *in* A_{ar}.

Hence, by the assumed completeness of F, $\sup A_{ar}$ *exists*. So we may define

$$a^r = \sup A_{ar}.$$[3]

We also put

$$a^{-r} = \frac{1}{a^r}.$$

If $0 < a < 1$ (so that $\frac{1}{a} > 1$), we put

$$a^r = \left(\frac{1}{a}\right)^{-r} \text{ and } a^{-r} = \frac{1}{a^r},$$

where

$$\left(\frac{1}{a}\right)^r = \sup A_{1/a,r},$$

as above.

Summing up, we have the following definitions.

Definition 2.

Given $a > 0$ in a complete field F, and $r \in E^1$, we define the following.

(i) If $r > 0$ and $a > 1$, then

$$a^r = \sup A_{ar} = \sup\{a^x \mid 0 < x \leq r,\ x \text{ rational}\}.$$

(ii) If $r > 0$ and $0 < a < 1$, then $a^r = \frac{1}{(1/a)^r}$, also written $(1/a)^{-r}$.

(iii) $a^{-r} = 1/a^r$. (This defines powers with *negative* exponents as well.)

[3] Note that, if r is a positive rational itself, then a^r is the *largest* a^x with $x \leq r$ (where a^r and a^x are as in Definition 1); thus $a^r = \max A_{ar} = \sup A_{ar}$, and so our present definition *agrees* with Definition 1. This excludes ambiguities.

We also define $0^r = 0$ for any real $r > 0$, and $a^0 = 1$ for any $a \in F$, $a \neq 0$; 0^0 *remains undefined*.

The power a^r is also defined if $a < 0$ and r is a *rational* $\frac{m}{n}$ with n *odd* because $a^r = \sqrt[n]{a^m}$ has sense in this case. (Why?) This does not work for other values of r. Therefore, in general, we assume $a > 0$.

Again, it is easy to show that the formulas in (1) remain also valid for powers with *real* exponents (see Problems 8–13 below), provided F is complete.

Problems on Roots, Powers, and Irrationals

The problems marked by $\Rightarrow$ are theoretically important. Study them!

1. Let $n \in N$ in E^1; let $p > 0$ and $a > 0$ be elements of an ordered field F. Prove that

(i) if $p^n > a$, then $(\exists\, x \in F)\ p > x > 0$ and $x^n > a$;

(ii) if $p^n < a$, then $(\exists\, x \in F)\ x > p$ and $x^n < a$.

[Hint: For (i), put

$$x = p - d, \text{ with } 0 < d < p.$$

Use the *Bernoulli inequality* (Problem 5(ii) in §§5–6) to find d such that

$$x^n = (p - d)^n > a,$$

i.e.,

$$\left(1 - \frac{d}{p}\right)^n > \frac{a}{p^n}.$$

Solving for d, show that this holds if

$$0 < d < \frac{p^n - a}{np^{n-1}} < p. \quad \text{(Why does such a } d \text{ exist?)}$$

For (ii), if $p^n < a$, then

$$\frac{1}{p^n} > \frac{1}{a}.$$

Use (i) with a and p replaced by $1/a$ and $1/p$.]

2. Prove Theorem 1 assuming that

(i) $a > 1$;

(ii) $0 < a < 1$ (the cases $a = 0$ and $a = 1$ are trivial).

[Hints: (i) Let

$$A = \{x \in F \mid x \geq 1,\ x^n > a\}.$$

Show that A is bounded below (by 1) and $A \neq \emptyset$ (e.g., $a + 1 \in A$—why?).

By completeness, put $p = \inf A$.

Then show that $p^n = a$ (i.e., p is the required $\sqrt[n]{a}$).

Indeed, if $p^n > a$, then Problem 1 would yield an $x \in A$ with

$$x < p = \inf A. \text{ (Contradiction!)}$$

Similarly, use Problem 1 to exclude $p^n < a$.

To prove *uniqueness*, use Problem 4(ii) of §§5–6.

Case (ii) reduces to (i) by considering $1/a$ instead of a.]

3. Prove Note 1.

[Hint: Suppose first that a is *not* divisible by any square of a prime, i.e.,

$$a = p_1 p_2 \cdots p_m,$$

where the p_k are *distinct* primes. (We assume it known that each $a \in N$ is the product of [possibly repeating] primes.) Then proceed as in the proof of Theorem 2, replacing "even" by "divisible by p_k."

The general case, $a = p^2 b$, reduces to the previous case since $\sqrt{a} = p\sqrt{b}$.]

4. Prove that if r is rational and q is not, then $r \pm q$ is irrational; so also are rq, q/r, and r/q if $r \neq 0$.

[Hint: Assume the opposite and find a contradiction.]

⇒5. Prove the *density of irrationals* in a complete field F: *If* $a < b$ $(a, b \in F)$, *there is an irrational* $x \in F$ with

$$a < x < b$$

(*hence infinitely many such irrationals* x). See also Chapter 1, §9, Problem 4.

[Hint: By Theorem 3 of §10,

$$(\exists\, r \in R) \quad a\sqrt{2} < r < b\sqrt{2},\ r \neq 0. \text{ (Why?)}$$

Put $x = r/\sqrt{2}$; see Problem 4].

6. Prove that the rational subfield R of any ordered field is Archimedean.

[Hint: If

$$x = \frac{k}{m} \text{ and } y = \frac{p}{q} \quad (k,\, m,\, p,\, q \in N),$$

then $nx > y$ for $n = mp + 1$].

7. Verify the formulas in (1) for powers with positive *rational* exponents r, s.

8. Prove that

(i) $a^{r+s} = a^r a^s$ and

(ii) $a^{r-s} = a^r / a^s$ for $r,\, s \in E^1$ and $a \in F$ $(a > 0)$.[4]

[Hints: For (i), if $r,\, s > 0$ and $a > 1$, use Problem 9 in §§8–9 to get

$$a^r a^s = \sup A_{ar} \sup A_{as} = \sup(A_{ar} A_{as}).$$

[4] In Problems 8–13, F is assumed *complete*. In a later chapter, we shall prove the formulas in (1) more simply. Thus the reader may as well omit their present verification. The problems are, however, useful as exercises.

Verify that

$$A_{ar}A_{as} = \{a^x a^y \mid x, y \in R,\ 0 < x \le r,\ 0 < y \le s\}$$
$$= \{a^z \mid z \in R,\ 0 < z \le r + s\} = A_{a,\, r+s}.$$

Hence deduce that

$$a^r a^s = \sup(A_{a,\, r+s}) = a^{r+s}$$

by Definition 2.

For (ii), if $r > s > 0$ and $a > 1$, then by (i),

$$a^{r-s}a^s = a^r;$$

so

$$a^{r-s} = \frac{a^r}{a^s}.$$

For the cases $r < 0$ or $s < 0$, or $0 < a < 1$, use the above results and Definition 2(ii)(iii).]

9. From Definition 2 prove that if $r > 0$ $(r \in E^1)$, then

$$a > 1 \Longleftrightarrow a^r > 1$$

for $a \in F$ $(a > 0)$.

10. Prove for $r, s \in E^1$ that

(i) $r < s \Leftrightarrow a^r < a^s$ if $a > 1$;

(ii) $r < s \Leftrightarrow a^r > a^s$ if $0 < a < 1$.

[Hints: (i) By Problems 8 and 9,

$$a^s = a^{r+(s-r)} = a^r a^{s-r} > a^r$$

since $a^{s-r} > 1$ if $a > 1$ and $s - r > 0$.

(ii) For the case $0 < a < 1$, use Definition 2(ii).]

11. Prove that

$$(a \cdot b)^r = a^r b^r \text{ and } \left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$$

for $r \in E^1$ and positive $a, b \in F$.
[Hint: Proceed as in Problem 8.]

12. Given $a, b > 0$ in F and $r \in E^1$, prove that

(i) $a > b \Leftrightarrow a^r > b^r$ if $r > 0$, and

(ii) $a > b \Leftrightarrow a^r < b^r$ if $r < 0$.

[Hint:

$$a > b \Longleftrightarrow \frac{a}{b} > 1 \Longleftrightarrow \left(\frac{a}{b}\right)^r > 1$$

if $r > 0$ by Problems 9 and 11].

13. Prove that
$$(a^r)^s = a^{rs}$$
for $r,\ s \in E^1$ and $a \in F$ $(a > 0)$.

[Hint: First let $r,\ s > 0$ and $a > 1$. To show that
$$(a^r)^s = a^{rs} = \sup A_{a,\,rs} = \sup\{a^{xy} \mid x, y \in R,\ 0 < xy \le rs\},$$
use Problem 13 in §§8–9. Thus prove that

(i) $(\forall\, x,\ y \in R \mid 0 < xy \le rs)\ a^{xy} \le (a^r)^s$, which is easy, and

(ii) $(\forall\, d > 1)\ (\exists\ x,\ y \in R \mid 0 < xy \le rs)\ (a^r)^s < da^{xy}$.

Fix any $d > 1$ and put $b = a^r$. Then
$$(a^r)^s = b^s = \sup A_{bs} = \sup\{b^y \mid y \in R,\ 0 < y \le s\}.$$
Hence there is some $y \in R$, $0 < y \le s$ such that
$$(a^r)^s < d^{\frac{1}{2}}(a^r)^y. \quad \text{(Why?)}$$
Fix that y. Now
$$a^r = \sup A_{ar} = \sup\{a^x \mid x \in R,\ 0 < x \le r\};$$
so
$$(\exists\ x \in R \mid 0 < x \le r) \quad a^r < d^{\frac{1}{2y}} a^x. \quad \text{(Why?)}$$
Combining all and using the formulas in (1) for *rationals* x, y, obtain
$$(a^r)^s < d^{\frac{1}{2}}(a^r)^y < d^{\frac{1}{2}}(d^{\frac{1}{2y}} a^x)^y = da^{xy},$$
thus proving (ii)].

§13. The Infinities. Upper and Lower Limits of Sequences

I. The Infinities. As we have seen, a set $A \neq \emptyset$ in E^1 has a lub (glb) if A is bounded above (respectively, below), *but not otherwise*.

In order to avoid this inconvenient restriction, we now add to E^1 two new objects *of arbitrary nature*, and call them "*minus infinity*" $(-\infty)$ and "*plus infinity*" $(+\infty)$, with the convention that $-\infty < +\infty$ and $-\infty < x < +\infty$ for all $x \in E^1$.

It is readily seen that with this convention, the laws of transitivity and trichotomy (Axioms VII and VIII) remain valid.

The set consisting of all reals and the two infinities is called the *extended real number system*. We denote it by E^* and call its elements *extended real numbers*. The ordinary reals are also called *finite numbers*, while $\pm\infty$ are the only two *infinite* elements of E^*. (*Caution*: They are *not* real numbers.)

At this stage we do not define any operations involving $\pm\infty$. (This will be done later.) However, the notions of upper and lower bound, maximum, minimum, supremum, and infimum *are* defined in E^* exactly as in E^1. In particular,

$$-\infty = \min E^* \text{ and } +\infty = \max E^*.$$

Thus in E^* *all sets are bounded*.

It follows that *in E^* every set $A \neq \emptyset$ has a lub and a glb*. For if A has none in E^1, it still has the upper bound $+\infty$ *in* E^*, which in this case is the *unique* (hence also the *least*) upper bound; thus $\sup A = +\infty$.[1] Similarly, $\inf A = -\infty$ if there is no other lower bound.[2] As is readily seen, *all properties of lub and glb stated in §§8–9 remain valid in* E^* (with the same proof). The only exception is Theorem 2(ii′) in the case $q = +\infty$ (respectively, $p = -\infty$) since $+\infty - \varepsilon$ and $-\infty + \varepsilon$ make no sense. Part (ii) of Theorem 2 *is* valid.

We can now define *intervals* in E^* exactly as in E^1 (§§8–9, Example (3)), allowing also infinite values of a, b, x. For example,

$$\begin{aligned}
(-\infty, a) &= \{x \in E^* \mid -\infty < x < a\} = \{x \in E^1 \mid x < a\};\\
(a, +\infty) &= \{x \in E^1 \mid a < x\};\\
(-\infty, +\infty) &= \{x \in E^* \mid -\infty < x < +\infty\} = E^1;\\
[-\infty, +\infty] &= \{x \in E^* \mid -\infty \le x \le +\infty\}; \text{ etc.}
\end{aligned}$$

Intervals with *finite* endpoints are said to be *finite*; all other intervals are called *infinite*. The infinite intervals

$$(-\infty, a),\ (-\infty, a],\ (a, +\infty),\ [a, +\infty), \quad a \in E^1,$$

are actually subsets of E^1, as is $(-\infty, +\infty)$. Thus we shall speak of infinite intervals *in* E^1 as well.

II. Upper and Lower Limits.[3] In Chapter 1, §§1–3 we already mentioned that a real number p is called the *limit* of a sequence $\{x_n\} \subseteq E^1$ ($p = \lim x_n$) iff

$$(\forall \varepsilon > 0)\ (\exists\, k)\ (\forall\, n > k) \quad |x_n - p| < \varepsilon, \text{ i.e., } p - \varepsilon < x_n < p + \varepsilon, \tag{1}$$

where $\varepsilon \in E^1$ and $n, k \in N$.

This may be stated as follows:

> For sufficiently large n ($n > k$), x_n becomes and *stays* as close to p as we like ("ε-close").

[1] This is true unless A consists of $-\infty$ alone, in which case $\sup A = -\infty$.
[2] It is also customary to define $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$. This is the *only* case where $\sup A < \inf A$.
[3] This topic may be deferred until Chapter 3, §14. It presupposes Chapter 1, §8.

We also define (in E^1 and E^*)

$$\lim_{n\to\infty} x_n = +\infty \Longleftrightarrow (\forall\, a \in E^1)\ (\exists\, k)\ (\forall\, n > k) \quad x_n > a \text{ and} \tag{2}$$

$$\lim_{n\to\infty} x_n = -\infty \Longleftrightarrow (\forall\, b \in E^1)\ (\exists\, k)\ (\forall\, n > k) \quad x_n < b. \tag{3}$$

Note that (2) and (3) make sense in E^1, too, since the symbols $\pm\infty$ do not occur on the right side of the formulas. Formula (2) means that x_n becomes *arbitrarily large* (larger than any $a \in E^1$ given in advance) for sufficiently large n ($n > k$). The interpretation of (3) is analogous. A more general and unified approach will now be developed for E^* (allowing *infinite terms* x_n, too).

Let $\{x_n\}$ be any sequence in E^*. For each n, let A_n be the set of all terms *from* x_n *onward*, i.e.,

$$\{x_n,\, x_{n+1},\, \ldots\}.$$

For example,

$$A_1 = \{x_1,\, x_2,\, \ldots\},\ A_2 = \{x_2,\, x_3,\, \ldots\},\ \text{etc.}$$

The A_n form a *contracting sequence* (see Chapter 1, §8) since

$$A_1 \supseteq A_2 \supseteq \cdots.$$

Now, for each n, let

$$p_n = \inf A_n \text{ and } q_n = \sup A_n,$$

also denoted

$$p_n = \inf_{k\ge n} x_k \text{ and } q_n = \sup_{k\ge n} x_k.$$

(These infima and suprema *always* exist in E^*, as noted above.) Since $A_n \supseteq A_{n+1}$, Corollary 2 of §§8–9 yields

$$\inf A_n \le \inf A_{n+1} \le \sup A_{n+1} \le \sup A_n.$$

Thus

$$p_1 \le p_2 \le \cdots \le p_n \le p_{n+1} \le \cdots \le q_{n+1} \le q_n \le \cdots \le q_2 \le q_1, \tag{4}$$

and so $\{p_n\}\uparrow$, while $\{q_n\}\downarrow$ in E^*. We also see that *each* q_m *is an upper bound of all* p_n and hence

$$q_m \ge \sup_n p_n \ (= \text{lub of all } p_n).$$

This, in turn, shows that this sup (call it $\underline{L}$) is a *lower bound of all* q_m, and so

$$\underline{L} \le \inf_m q_m.$$

We put

$$\inf_m q_m = \overline{L}.$$

Definition 1.

For each sequence $\{x_n\} \subseteq E^*$, we define its *upper limit* $\overline{L}$ and its *lower limit* $\underline{L}$, denoted

$$\overline{L} = \overline{\lim}\, x_n = \limsup_{n\to\infty} x_n \text{ and } \underline{L} = \underline{\lim}\, x_n = \liminf_{n\to\infty} x_n,$$

as follows.

We put $(\forall\, n)$

$$q_n = \sup_{k\geq n} x_k \text{ and } p_n = \inf_{k\geq n} x_k,$$

as before. Then we set

$$\overline{L} = \overline{\lim}\, x_n = \inf_n q_n \text{ and } \underline{L} = \underline{\lim}\, x_n = \sup_n p_n, \text{ all in } E^*. \tag{4}$$

Here and below, $\inf_n q_n$ is the inf of *all* q_n, and $\sup_n p_n$ is the sup of *all* p_n.

Corollary 1. *For any sequence in* E^**,*

$$\inf_n x_n \leq \underline{\lim}\, x_n \leq \overline{\lim}\, x_n \leq \sup_n x_n.$$

For, as we noted above,

$$\underline{L} = \sup_n p_n \leq \inf_m q_m = \overline{L}.$$

Also,

$$\underline{L} \geq p_n = \inf A_n \geq \inf A_1 = \inf_n x_n \text{ and}$$
$$\overline{L} \leq q_n = \sup A_n \leq \sup A_1 = \sup_n x_n,$$

with A_n as above.

Examples.

(a) Let

$$x_n = \frac{1}{n}.$$

Here

$$q_1 = \sup\left\{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\} = 1,\ q_2 = \frac{1}{2},\ q_n = \frac{1}{n}.$$

Hence

$$\overline{L} = \inf_n q_n = \inf\left\{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\} = 0,$$

as easily follows by Theorem 2 in §§8–9 and the Archimedean property. (Verify!) Also,

$$p_1 = \inf_{k\geq 1} \frac{1}{k} = 0,\ p_2 = \inf_{k\geq 2} \frac{1}{k} = 0,\ \ldots,\ p_n = \inf_{k\geq n} \frac{1}{k} = 0.$$

Since all p_n are 0, so is $\overline{L} = \sup_n p_n$. Thus here $\underline{L} = \overline{L} = 0$.

(b) Consider the sequence

$$1,\ -1,\ 2,\ -\frac{1}{2},\ \ldots,\ n,\ -\frac{1}{n},\ \ldots.$$

Here

$$p_1 = -1 = p_2,\ p_3 = -\frac{1}{2} = p_4,\ \ldots;\ p_{2n-1} = -\frac{1}{n} = p_{2n}.$$

Thus

$$\underline{\lim}\, x_n = \sup_n p_n = \sup\left\{-1,\ -\frac{1}{2},\ \ldots,\ -\frac{1}{n},\ \ldots\right\} = 0.$$

On the other hand, $q_n = +\infty$ for all n. (Why?) Thus

$$\overline{\lim}\, x_n = \inf_n q_n = +\infty.$$

Theorem 1.

(i) *If $x_n \geq b$ for infinitely many n, then*

$$\overline{\lim}\, x_n \geq b \quad \textit{as well.}$$

(ii) *If $x_n \leq a$ for all but finitely many n,*[4] *then*

$$\overline{\lim}\, x_n \leq a \quad \textit{as well.}$$

Similarly for lower limits (*with all inequalities reversed*).

Proof.

(i) If $x_n \geq b$ for infinitely many n, then such n must occur in *each* set

$$A_m = \{x_m,\ x_{m+1},\ \ldots\}.$$

Hence

$$(\forall\, m) \quad q_m = \sup A_m \geq b;$$

so $\overline{L} = \inf_m q_m \geq b$, by Corollary 1 of §§8–9.

(ii) If $x_n \leq a$ except *finitely many* n, let n_0 be the *last* of these "exceptional" values of n.

Then for $n > n_0$, $x_n \leq a$, i.e., the set

$$A_n = \{x_n,\ x_{n+1},\ \ldots\}$$

[4] In other words, for all except (at most) a finite number of terms x_n. This is stronger than just "*infinitely many n*" (allowing infinitely many *exceptions* as well). *Caution*: Avoid confusing "*all but finitely many*" with just "*infinitely many*."

is bounded above by a; so

$$(\forall\, n > n_0) \quad q_n = \sup A_n \le a.$$

Hence certainly $\overline{L} = \inf_n q_n \le a$. □

Corollary 2.

(i) *If* $\overline{\lim}\, x_n > a$, *then* $x_n > a$ *for infinitely many* n.

(ii) *If* $\overline{\lim}\, x_n < b$, *then* $x_n < b$ *for all but finitely many* n.

Similarly for lower limits (*with all inequalities reversed*).

Proof. Assume the opposite and find a contradiction to Theorem 1. □

To unify our definitions, we now introduce some useful notions.

By a *neighborhood of* p, briefly G_p,[5] we mean, for $p \in E^1$, any interval of the form

$$(p - \varepsilon,\, p + \varepsilon), \quad \varepsilon > 0.$$

If $p = +\infty$ (respectively, $p = -\infty$), G_p is an infinite interval of the form

$$(a,\, +\infty] \text{ (respectively, } [-\infty,\, b)), \text{ with } a,\, b \in E^1.$$

We can now combine formulas (1)–(3) into one equivalent definition.

Definition 2.

An element $p \in E^*$ (finite or not) is called the *limit* of a sequence $\{x_n\}$ in E^* iff each G_p (no matter how small it is) contains all but finitely many x_n, i.e. all x_n from some x_k onward. In symbols,

$$(\forall\, G_p)\ (\exists\, k)\ (\forall\, n > k) \quad x_n \in G_p. \tag{5}$$

We shall use the notation

$$p = \lim x_n \text{ or } \lim_{n\to\infty} x_n.$$

Indeed, if $p \in E^1$, then $x_n \in G_p$ means

$$p - \varepsilon < x_n < p + \varepsilon,$$

as in (1). If, however, $p = \pm\infty$, it means

$$x_n > a \text{ (respectively, } x_n < b),$$

as in (2) and (3).

[5] This terminology and notation anticipates some more general ideas in Chapter 3, §11.

Theorem 2. *We have* $q = \overline{\lim}\, x_n$ *in* E^* *iff*

(i′) *each neighborhood* G_q *contains* x_n *for infinitely many* n, *and*

(ii′) *if* $q < b$, *then* $x_n \geq b$ *for at most finitely many* n.[6]

Proof. If $q = \overline{\lim}\, x_n$, Corollary 2 yields (ii′).

It also shows that any interval (a, b), with $a < q < b$, contains *infinitely many* x_n (for there are infinitely many $x_n > a$, and only finitely many $x_n \geq b$, by (ii′)).

Now if $q \in E^1$,

$$G_q = (q - \varepsilon,\, q + \varepsilon)$$

is such an interval, so we obtain (i′). The cases $q = \pm\infty$ are analogous; we leave them to the reader.

Conversely, assume (i′) and (ii′).

Seeking a contradiction, let $q < \overline{L}$; say,

$$q < b < \overline{\lim}\, x_n.$$

Then Corollary 2(i) yields $x_n > b$ for infinitely many n, contrary to our assumption (ii′).

Similarly, $q > \overline{\lim}\, x_n$ would contradict (i′).

Thus necessarily $q = \overline{\lim}\, x_n$. $\square$

Theorem 3. *We have* $q = \lim x_n$ *in* E^* *iff*

$$\underline{\lim}\, x_n = \overline{\lim}\, x_n = q.$$

Proof. Suppose

$$\underline{\lim}\, x_n = \overline{\lim}\, x_n = q.$$

If $q \in E^1$, then every G_q is an interval (a, b), $a < q < b$; therefore, Corollary 2(ii) and its analogue for $\underline{\lim}\, x_n$ imply (with q treated as *both* $\overline{\lim}\, x_n$ and $\underline{\lim}\, x_n$) that

$$a < x_n < b \quad \text{for } \textit{all but finitely many } n.$$

Thus by Definition 2, $q = \lim x_n$, as claimed.

Conversely, if so, then any G_q (no matter how small) contains all but finitely many x_n. Hence *so does any interval* (a, b) *with* $a < q < b$, for it contains some small G_q.

Now, exactly as in the proof of Theorem 2, one *excludes*

$$q \neq \underline{\lim}\, x_n \text{ and } q \neq \overline{\lim}\, x_n.$$

This settles the case $q \in E^1$. The cases $q = \pm\infty$ are quite analogous. $\square$

[6] A similar theorem (with all inequalities reversed) holds for $\underline{\lim}\, x_n$.

Problems on Upper and Lower Limits of Sequences in E^*

1. Complete the missing details in the proofs of Theorems 2 and 3, Corollary 1, and Examples (a) and (b).

2. State and prove the analogues of Theorems 1 and 2 and Corollary 2 for $\underline{\lim}\, x_n$.

3. Find $\overline{\lim}\, x_n$ and $\underline{\lim}\, x_n$ if

(a) $x_n = c$ (constant);

(b) $x_n = -n$;

(c) $x_n = n$; and

(d) $x_n = (-1)^n n - n$.

Does $\lim x_n$ exist in each case?

⇒4. A sequence $\{x_n\}$ is said to *cluster* at $q \in E^*$, and q is called its *cluster point*, iff each G_q contains x_n for infinitely many values of n.

Show that both $\underline{L}$ and $\overline{L}$ are cluster points ($\underline{L}$ the *least* and $\overline{L}$ the *largest*).

[Hint: Use Theorem 2 and its analogue for $\underline{L}$.

To show that no $p < \underline{L}$ (or $q > \overline{L}$) is a cluster point, assume the opposite and find a contradiction to Corollary 2.]

⇒5. Prove that

(i) $\overline{\lim}(-x_n) = -\underline{\lim}\, x_n$ and

(ii) $\overline{\lim}(ax_n) = a \cdot \overline{\lim}\, x_n$ if $0 \le a < +\infty$.

6. Prove that

$$\overline{\lim}\, x_n < +\infty \ (\underline{\lim}\, x_n > -\infty)$$

iff $\{x_n\}$ is bounded above (below) in E^1.

7. Prove that if $\{x_n\}$ and $\{y_n\}$ are *bounded in* E^1, then

$$\begin{aligned}\overline{\lim}\, x_n + \overline{\lim}\, y_n &\ge \overline{\lim}(x_n + y_n) \ge \overline{\lim}\, x_n + \underline{\lim}\, y_n \\ &\ge \underline{\lim}(x_n + y_n) \ge \underline{\lim}\, x_n + \underline{\lim}\, y_n.\end{aligned}$$

[Hint: Prove the first inequality and then use that and Problem 5(i) for the others.]

⇒8. Prove that if $p = \lim x_n$ *in* E^1, then

$$\underline{\lim}(x_n + y_n) = p + \underline{\lim}\, y_n;$$

similarly for $\overline{L}$.

⇒9. Prove that if $\{x_n\}$ is monotone, then $\lim x_n$ exists *in* E^*. Specifically, if $\{x_n\}\uparrow$, then

$$\lim x_n = \sup_n x_n,$$

and if $\{x_n\}\downarrow$, then

$$\lim x_n = \inf_n x_n.$$

⇒**10.** Prove that

(i) if $\lim x_n = +\infty$ and $(\forall n)\ x_n \le y_n$, then also $\lim y_n = +\infty$, and

(ii) if $\lim x_n = -\infty$ and $(\forall n)\ y_n \le x_n$, then also $\lim y_n = -\infty$.

11. Prove that if $x_n \le y_n$ for all n, then

$$\underline{\lim}\, x_n \le \underline{\lim}\, y_n \text{ and } \overline{\lim}\, x_n \le \overline{\lim}\, y_n.$$

Chapter 3

Vector Spaces. Metric Spaces

§§1–3. The Euclidean n-Space, E^n

By definition, the *Euclidean n-space* E^n is the set of all possible ordered n-tuples of real numbers, i.e., the Cartesian product

$$E^1 \times E^1 \times \cdots \times E^1 \ (n \text{ times}).$$

In particular, $E^2 = E^1 \times E^1 = \{(x, y) \mid x, y \in E^1\}$,

$$E^3 = E^1 \times E^1 \times E^1 = \{(x, y, z) \mid x, y, z \in E^1\},$$

and so on. E^1 itself is a special case of E^n $(n = 1)$.

In a familiar way, pairs (x, y) can be plotted as *points* of the xy-plane, or as "*vectors*" (directed line segments) joining $(0, 0)$ to such points. Therefore, the pairs (x, y) themselves are called *points* or *vectors* in E^2; similarly for E^3.

In E^n $(n > 3)$, there is no actual geometric representation, but it is convenient to use *geometric language* in this case, too. Thus any ordered n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers will also be called a *point* or *vector* in E^n, and the single numbers $x_1, x_2, \ldots, x_n$ are called its *coordinates* or *components*. A point in E^n is often denoted by a *single* letter (preferably with a bar or an arrow above it), and then its n components are denoted by *the same* letter, with subscripts (but without the bar or arrow). For example,

$$\bar{x} = (x_1, \ldots, x_n), \ \vec{u} = (u_1, \ldots, u_n), \text{ etc.};$$

$\bar{x} = (0, -1, 2, 4)$ is a point (vector) in E^4 with coordinates $0, -1, 2$, and 4 (in this order). The formula $\bar{x} \in E^n$ means that $\bar{x} = (x_1, \ldots, x_n)$ is a point (vector) in E^n. Since such "points" are *ordered* n-tuples, $\bar{x}$ and $\bar{y}$ are equal ($\bar{x} = \bar{y}$) iff the *corresponding* coordinates are the same, i.e., $x_1 = y_1$, $x_2 = y_2$, $\ldots$, $x_n = y_n$ (see Problem 1 below).

The point whose coordinates are all 0 is called the *zero-vector* or the *origin*, denoted $\vec{0}$ or $\bar{0}$. The vector whose kth component is 1, and the other components are 0, is called the kth *basic unit vector*, denoted $\vec{e}_k$. There are exactly n such vectors,

$$\vec{e}_1 = (1, 0, 0, \ldots, 0),\ \vec{e}_2 = (0, 1, 0, \ldots, 0),\ \ldots,\ \vec{e}_n = (0, \ldots, 0, 1).$$

In E^3, we often write $\bar{i}$, $\bar{j}$, and $\bar{k}$ for $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, and (x, y, z) for (x_1, x_2, x_3). Similarly in E^2. Single real numbers are called *scalars* (as opposed to *vectors*).

Definitions.

Given $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_n)$ in E^n, we define the following.

1. The *sum* of $\bar{x}$ and $\bar{y}$,

$$\bar{x} + \bar{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \text{ (hence } \bar{x} + \bar{0} = \bar{x}).$$[1]

2. The *dot product*, or *inner product*, of $\bar{x}$ and $\bar{y}$,

$$\bar{x} \cdot \bar{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

3. The *distance* between $\bar{x}$ and $\bar{y}$,

$$\rho(\bar{x}, \bar{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}\,.$$

4. The *absolute value*, or *length*, of $\bar{x}$,

$$|\bar{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \rho(\bar{x}, \bar{0}) = \sqrt{\bar{x} \cdot \bar{x}}$$

(three formulas that are all equal by Definitions 2 and 3).

5. The *inverse* of $\bar{x}$,

$$-\bar{x} = (-x_1, -x_2, \ldots, -x_n).$$

6. The *product* of $\bar{x}$ by a *scalar* $c \in E^1$,

$$c\bar{x} = \bar{x}c = (cx_1, cx_2, \ldots, cx_n);$$

in particular, $(-1)\bar{x} = (-x_1, -x_2, \ldots, -x_n) = -\bar{x}$, $1\bar{x} = \bar{x}$, and $0\bar{x} = \bar{0}$.

7. The *difference* of $\bar{x}$ and $\bar{y}$,

$$\bar{x} - \bar{y} = \overrightarrow{yx} = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n).$$

In particular, $\bar{x} - \bar{0} = \bar{x}$ and $\bar{0} - \bar{x} = -\bar{x}$. (Verify!)

Note 1. Definitions 2–4 yield *scalars*, while the rest are *vectors*.

Note 2. We shall not define *inequalities* ($<$) in E^n $(n \geq 2)$, nor shall we define vector products other than the *dot product* (2), which is a *scalar*. (However, cf. §8.)

[1] Sums of three or more vectors are defined by induction, as in Chapter 2, §§5–6.

Note 3. From Definitions 3, 4, and 7, we obtain $\rho(\bar{x}, \bar{y}) = |\bar{x} - \bar{y}|$. (Verify!)

Note 4. We often write $\bar{x}/c$ for $(1/c)\bar{x}$, where $c \in E^1$, $c \neq 0$.

Note 5. In E^1, $\bar{x} = (x_1) = x_1$. Thus, by Definition 4,

$$|\bar{x}| = \sqrt{x_1^2} = |x_1|,$$

where $|x_1|$ is defined as in Chapter 2, §§1–4, Definition 4. Thus the two definitions *agree*.

We call $\bar{x}$ a *unit vector* iff its length is 1, i.e., $|x| = 1$. Note that if $\bar{x} \neq \bar{0}$, then $\bar{x}/|\bar{x}|$ is a unit vector, since

$$\left|\frac{\bar{x}}{|\bar{x}|}\right| = \sqrt{\frac{x_1^2}{|\bar{x}|^2} + \cdots + \frac{x_n^2}{|\bar{x}|^2}} = 1.$$

The vectors $\bar{x}$ and $\bar{y}$ are said to be *orthogonal* or *perpendicular* ($\bar{x} \perp \bar{y}$) iff $\bar{x} \cdot \bar{y} = 0$ and *parallel* ($\bar{x} \parallel \bar{y}$) iff $\bar{x} = t\bar{y}$ or $\bar{y} = t\bar{x}$ for some $t \in E^1$. Note that $\bar{x} \perp \bar{0}$ and $\bar{x} \parallel \bar{0}$.

Examples.

If $\bar{x} = (0, -1, 4, 2)$ and $\bar{y} = (2, 2, -3, 2)$ are vectors in E^4, then

$$\begin{aligned} \bar{x} + \bar{y} &= (2, 1, 1, 4); \\ \bar{x} - \bar{y} &= (-2, -3, 7, 0); \\ \rho(\bar{x}, \bar{y}) = |\bar{x} - \bar{y}| &= \sqrt{2^2 + 3^2 + 7^2 + 0^2} = \sqrt{62}\,; \\ (\bar{x} + \bar{y}) \cdot (\bar{x} - \bar{y}) &= 2(-2) + 1(-3) + 7 + 0 = 0. \end{aligned}$$

So $(\bar{x} + \bar{y}) \perp (\bar{x} - \bar{y})$ here.

Theorem 1. *For any vectors $\bar{x}$, $\bar{y}$, and $\bar{z} \in E^n$ and any a, $b \in E^1$, we have*

(a) $\bar{x} + \bar{y}$ *and* $a\bar{x}$ *are vectors in* E^n (*closure laws*);

(b) $\bar{x} + \bar{y} = \bar{y} + \bar{x}$ (*commutativity of vector addition*);

(c) $(\bar{x} + \bar{y}) + \bar{z} = \bar{x} + (\bar{y} + \bar{z})$ (*associativity of vector addition*);

(d) $\bar{x} + \bar{0} = \bar{0} + \bar{x} = \bar{x}$, *i.e.*, $\bar{0}$ *is the neutral element of addition*;

(e) $\bar{x} + (-\bar{x}) = \bar{0}$, *i.e.*, $-\bar{x}$ *is the additive inverse of* $\bar{x}$;

(f) $a(\bar{x} + \bar{y}) = a\bar{x} + a\bar{y}$ *and* $(a + b)\bar{x} = a\bar{x} + b\bar{x}$ (*distributive laws*);

(g) $(ab)\bar{x} = a(b\bar{x})$;

(h) $1\bar{x} = \bar{x}$.

Proof. Assertion (a) is immediate from Definitions 1 and 6. The rest follows from corresponding properties of *real* numbers.

For example, to prove (b), let $\bar{x} = (x_1, \ldots, x_n)$, $\bar{y} = (y_1, \ldots, y_n)$. Then by definition, we have

$$\bar{x} + \bar{y} = (x_1 + y_1, \ldots, x_n + y_n) \text{ and } \bar{y} + \bar{x} = (y_1 + x_1, \ldots, y_n + x_n).$$

The right sides in both expressions, however, coincide since addition is commutative *in* E^1. Thus $\bar{x} + \bar{y} = \bar{y} + \bar{x}$, as claimed; similarly for the rest, which we leave to the reader. □

Theorem 2. *If $\bar{x} = (x_1, \ldots, x_n)$ is a vector in E^n, then, with $\bar{e}_k$ as above,*

$$\bar{x} = x_1\bar{e}_1 + x_2\bar{e}_2 + \cdots + x_n\bar{e}_n = \sum_{k=1}^{n} x_k\bar{e}_k.$$

Moreover, if $\bar{x} = \sum_{k=1}^{n} a_k\bar{e}_k$ for some $a_k \in E^1$, then necessarily $a_k = x_k$, $k = 1, \ldots, n$.

Proof. By definition,

$$\bar{e}_1 = (1, 0, \ldots, 0),\ \bar{e}_2 = (0, 1, \ldots, 0),\ \ldots,\ \bar{e}_n = (0, 0, \ldots, 1).$$

Thus

$$x_1\bar{e}_1 = (x_1, 0, \ldots, 0), x_2\bar{e}_2 = (0, x_2, \ldots, 0), \ldots, x_n\bar{e}_n = (0, 0, \ldots, x_n).$$

Adding up componentwise, we obtain

$$\sum_{k=1}^{n} x_k\bar{e}_k = (x_1, x_2, \ldots, x_n) = \bar{x},$$

as asserted.

Moreover, if the x_k are replaced by any other $a_k \in E^1$, the same process yields

$$(a_1, \ldots, a_n) = \bar{x} = (x_1, \ldots, x_n),$$

i.e., *the two n-tuples coincide*, whence $a_k = x_k$, $k = 1, \ldots, n$. □

Note 6. Any sum of the form

$$\sum_{k=1}^{m} a_k\bar{x}_k \quad (a_k \in E^1,\ \bar{x}_k \in E^n)$$

is called a *linear combination* of the vectors $\bar{x}_k$ (whose number m is *arbitrary*). Thus Theorem 2 shows that *any $\bar{x} \in E^n$ can be expressed, in a unique way, as a linear combination of the n basic unit vectors*. In E^3, we write

$$\bar{x} = x_1\bar{i} + x_2\bar{j} + x_3\bar{k}.$$

Note 7. If, as above, some vectors are *numbered* (e.g., $\bar{x}_1$, $\bar{x}_2$, $\ldots$, $\bar{x}_m$), we denote their components by attaching a *second* subscript; for example, the components of $\bar{x}_1$ are $x_{11}, x_{12}, \ldots, x_{1n}$.

Theorem 3. *For any vectors $\bar{x}$, $\bar{y}$, and $\bar{z} \in E^n$ and any $a, b \in E^1$, we have*

(a) $\bar{x} \cdot \bar{x} \geq 0$, *and* $\bar{x} \cdot \bar{x} > 0$ *iff* $\bar{x} \neq \bar{0}$;

(b) $(a\bar{x}) \cdot (b\bar{y}) = (ab)(\bar{x} \cdot \bar{y})$;

(c) $\bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x}$ (*commutativity of inner products*);

(d) $(\bar{x} + \bar{y}) \cdot \bar{z} = \bar{x} \cdot \bar{z} + \bar{y} \cdot \bar{z}$ (*distributive law*).

Proof. To prove these properties, express all in terms of the components of $\bar{x}$, $\bar{y}$, and $\bar{z}$, and proceed as in Theorem 1. $\square$

Note that (b) implies $\bar{x} \cdot \bar{0} = 0$ (put $a = 1$, $b = 0$).

Theorem 4. *For any vectors $\bar{x}$ and $\bar{y} \in E^n$ and any $a \in E^1$, we have the following properties:*

(a′) $|\bar{x}| \geq 0$, *and* $|\bar{x}| > 0$ *iff* $\bar{x} \neq \bar{0}$.

(b′) $|a\bar{x}| = |a||\bar{x}|$.

(c′) $|\bar{x} \cdot \bar{y}| \leq |\bar{x}|\,|\bar{y}|$, *or, in components,*

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right) \quad (\textit{Cauchy–Schwarz inequality}).$$

Equality, $|\bar{x} \cdot \bar{y}| = |\bar{x}|\,|\bar{y}|$, *holds iff* $\bar{x} \parallel \bar{y}$.

(d′) $|\bar{x} + \bar{y}| \leq |\bar{x}| + |\bar{y}|$ *and* $\big||\bar{x}| - |\bar{y}|\big| \leq |\bar{x} - \bar{y}|$ (*triangle inequalities*).

Proof. Property (a′) follows from Theorem 3(a) since

$$|\bar{x}|^2 = \bar{x} \cdot \bar{x} \text{ (see Definition 4).}$$

For (b′), use Theorem 3(b), to obtain

$$(a\bar{x}) \cdot (a\bar{x}) = a^2(\bar{x} \cdot \bar{x}) = a^2|\bar{x}|^2.$$

By Definition 4, however,

$$(a\bar{x}) \cdot (a\bar{x}) = |a\bar{x}|^2.$$

Thus

$$|a\bar{x}|^2 = a^2|x|^2$$

so that $|a\bar{x}| = |a||\bar{x}|$, as claimed.

Now we prove (c′). If $\bar{x} \parallel \bar{y}$ then $\bar{x} = t\bar{y}$ or $\bar{y} = t\bar{x}$; so $|\bar{x} \cdot \bar{y}| = |\bar{x}||\bar{y}|$ follows by (b′). (Verify!)

Otherwise, $\bar{x} \neq t\bar{y}$ and $\bar{y} \neq t\bar{x}$ for all $t \in E^1$. Then we obtain, *for all* $t \in E^1$,

$$0 \neq |t\bar{x} - \bar{y}|^2 = \sum_{k=1}^{n} (tx_k - y_k)^2 = t^2 \sum_{k=1}^{n} x_k^2 - 2t \sum_{k=1}^{n} x_k y_k + \sum_{k=1}^{n} y_k^2.$$

Thus, setting

$$A = \sum_{k=1}^{n} x_k^2, \ B = 2\sum_{k=1}^{n} x_k y_k, \text{ and } C = \sum_{k=1}^{n} y_k^2,$$

we see that the quadratic equation

$$0 = At^2 - Bt + C$$

has *no* real solutions in t, so its discriminant, $B^2 - 4AC$, must be *negative*; i.e.,

$$4\left(\sum_{k=1}^{n} x_k y_k\right)^2 - 4\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right) < 0,$$

proving (c′).

To prove (d′), use Definition 2 and Theorem 3(d), to obtain

$$|\bar{x} + \bar{y}|^2 = (\bar{x} + \bar{y}) \cdot (\bar{x} + \bar{y}) = \bar{x} \cdot \bar{x} + \bar{y} \cdot \bar{y} + 2\bar{x} \cdot \bar{y} = |\bar{x}|^2 + |\bar{y}|^2 + 2\bar{x} \cdot \bar{y}.$$

But $\bar{x} \cdot \bar{y} \leq |\bar{x}|\,|\bar{y}|$ by (c′). Thus we have

$$|\bar{x} + \bar{y}|^2 \leq |\bar{x}|^2 + |\bar{y}|^2 + 2|\bar{x}|\,|\bar{y}| = (|\bar{x}| + |\bar{y}|)^2,$$

whence $|\bar{x} + \bar{y}| \leq |\bar{x}| + |\bar{y}|$, as required.

Finally, replacing here $\bar{x}$ by $\bar{x} - \bar{y}$, we have

$$|\bar{x} - \bar{y}| + |\bar{y}| \geq |\bar{x} - \bar{y} + \bar{y}| = |\bar{x}|, \text{ or } |\bar{x} - \bar{y}| \geq |\bar{x}| - |\bar{y}|.$$

Similarly, replacing $\bar{y}$ by $\bar{y} - \bar{x}$, we get $|\bar{x} - \bar{y}| \geq |\bar{y}| - |\bar{x}|$. Hence

$$|\bar{x} - \bar{y}| \geq \pm\big(|\bar{x}| - |\bar{y}|\big),$$

i.e., $|\bar{x} - \bar{y}| \geq \big||\bar{x}| - |\bar{y}|\big|$, proving the second formula in (d′). □

Theorem 5. *For any points* $\bar{x}$, $\bar{y}$, *and* $\bar{z} \in E^n$, *we have*

(i) $\rho(\bar{x}, \bar{y}) \geq 0$, *and* $\rho(\bar{x}, \bar{y}) = 0$ *iff* $\bar{x} = \bar{y}$;

(ii) $\rho(\bar{x}, \bar{y}) = \rho(\bar{y}, \bar{x})$;

(iii) $\rho(\bar{x}, \bar{z}) \leq \rho(\bar{x}, \bar{y}) + \rho(\bar{y}, \bar{z})$ (*triangle inequality*).

Proof.

(i) By Definition 3 and Note 3, $\rho(\bar{x}, \bar{y}) = |\bar{x} - \bar{y}|$; therefore, by Theorem 4(a′), $\rho(\bar{x}, \bar{y}) = |\bar{x} - \bar{y}| \geq 0$.

Also, $|\bar{x} - \bar{y}| > 0$ iff $\bar{x} - \bar{y} \neq 0$, i.e., iff $\bar{x} \neq \bar{y}$. Hence $\rho(\bar{x}, \bar{y}) \neq 0$ iff $\bar{x} \neq \bar{y}$, and assertion (i) follows.

(ii) By Theorem 4(b′), $|\bar{x} - \bar{y}| = |(-1)(\bar{y} - \bar{x})| = |\bar{y} - \bar{x}|$, so (ii) follows.

(iii) By Theorem 4(d′),

$$\rho(\bar{x}, \bar{y}) + \rho(\bar{y}, \bar{z}) = |\bar{x} - \bar{y}| + |\bar{y} - \bar{z}| \geq |\bar{x} - \bar{y} + \bar{y} - \bar{z}| = \rho(\bar{x}, \bar{z}). \quad \square$$

Note 8. We also have $|\rho(\bar{x}, \bar{y}) - \rho(\bar{z}, \bar{y})| \leq \rho(\bar{x}, \bar{z})$. (Prove it!) The two triangle inequalities have a simple geometric interpretation (which explains their name). If $\bar{x}$, $\bar{y}$, and $\bar{z}$ are treated as the vertices of a triangle, we obtain that the length of a side, $\rho(\bar{x}, \bar{z})$ never exceeds the sum of the two other sides and is never less than their difference.

As E^1 is a special case of E^n (in which "vectors" are *single* numbers), all our theory applies to E^1 as well. In particular, distances in E^1 are defined by $\rho(x, y) = |x - y|$ and obey the three laws of Theorem 5. Dot products in E^1 become *ordinary* products xy. (Why?) From Theorems 4(b′)(d′), we have

$$|a|\,|x| = |ax|; \;\; |x + y| \leq |x| + |y|; \;\; |x - y| \geq \big||x| - |y|\big| \quad (a, x, y \in E^1).$$

Problems on Vectors in E^n

1. Prove by induction on n that

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \text{ iff } x_k = y_k, \; k = 1, 2, \ldots, n.$$

[Hint: Use Problem 6(ii) of Chapter 1, §§1–3, and Example (i) in Chapter 2, §§5-6.]

2. Complete the proofs of Theorems 1 and 3 and Notes 3 and 8.

3. Given $\bar{x} = (-1, 2, 0, -7)$, $\bar{y} = (0, 0, -1, -2)$, and $\bar{z} = (2, 4, -3, -3)$ in E^4, express $\bar{x}$, $\bar{y}$, and $\bar{z}$ as linear combinations of the basic unit vectors. Also, compute their absolute values, their inverses, as well as their mutual sums, differences, dot products, and distances. Are any of them orthogonal? Parallel?

4. With $\bar{x}$, $\bar{y}$, and $\bar{z}$ as in Problem 3, find scalars a, b, and c such that

$$a\bar{x} + b\bar{y} + c\bar{z} = \bar{u},$$

when

(i) $\bar{u} = \bar{e}_1$; (ii) $\bar{u} = \bar{e}_3$;

(iii) $\bar{u} = (-2, 4, 0, 1)$; (iv) $\bar{u} = \bar{0}$.

5. A finite set of vectors $\bar{x}, \bar{x}_2, \ldots, \bar{x}_m$ is said to be *dependent* iff there are scalars $a_1, \ldots, a_m$, *not all zero*, such that

$$\sum_{k=1}^{m} a_k \bar{x}_k = \bar{0},$$

and *independent* otherwise. Prove the independence of the following sets of vectors:

(a) $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ in E^n;

(b) $(1, 2, -3, 4)$ and $(2, 3, 0, 0)$ in E^4;

(c) $(2, 0, 0)$, $(4, -1, 3)$, and $(0, 4, 1)$ in E^3;

(d) the vectors $\bar{x}$, $\bar{y}$, and $\bar{z}$ of Problem 3.

6. Prove (for E^2 and E^3) that

$$\bar{x} \cdot \bar{y} = |\bar{x}|\,|\bar{y}| \cos\alpha,$$

where α is the angle between the vectors $\overrightarrow{0x}$ and $\overrightarrow{0y}$; we denote α by $\langle \bar{x}, \bar{y} \rangle$.

[Hint: Consider the triangle $\bar{0}\bar{x}\bar{y}$, with sides $\bar{x} = \overrightarrow{0x}$, $\bar{y} = \overrightarrow{0y}$, and $\overrightarrow{xy} = \vec{y} - \vec{x}$ (see Definition 7). By the law of cosines,

$$|\vec{x}|^2 + |\vec{y}|^2 - 2|\vec{x}|\,|\vec{y}| \cos\alpha = |\vec{y} - \vec{x}|^2.$$

Now substitute $|\vec{x}|^2 = \vec{x} \cdot \vec{x}$, $|\vec{y}|^2 = \vec{y} \cdot \vec{y}$, and

$$|\vec{y} - \vec{x}|^2 = (\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x}) = \vec{y} \cdot \vec{y} + \vec{x} \cdot \vec{x} - 2\vec{x} \cdot \vec{y}. \text{ (Why?)}$$

Then simplify.]

7. Motivated by Problem 6, define in E^n

$$\langle \bar{x}, \bar{y} \rangle = \arccos \frac{\bar{x} \cdot \bar{y}}{|\bar{x}|\,|\bar{y}|} \text{ if } \bar{x} \text{ and } \bar{y} \text{ are nonzero.}$$

(Why does an angle with such a cosine exist?) Prove that

(i) $\bar{x} \perp \bar{y}$ iff $\cos\langle \bar{x}, \bar{y} \rangle = 0$, i.e., $\langle \bar{x}, \bar{y} \rangle = \dfrac{\pi}{2}$;

(ii) $\displaystyle\sum_{k=1}^{n} \cos^2\langle \bar{x}, \bar{e}_k \rangle = 1.$

8. Continuing Problems 3 and 7, find the cosines of the angles between the *sides*, $\overrightarrow{xy}$, $\overrightarrow{yz}$, and $\overrightarrow{zx}$ of the triangle $\bar{x}\bar{y}\bar{z}$, with $\bar{x}$, $\bar{y}$, and $\bar{z}$ as in Problem 3.

9. Find a unit vector in E^4, with positive components, that forms equal angles with the axes, i.e., with the basic unit vectors (see Problem 7).

10. Prove for E^n that if $\bar{u}$ is orthogonal to each of the basic unit vectors $\bar{e}_1$, $\bar{e}_2, \ldots, \bar{e}_n$, then $\bar{u} = \bar{0}$. Deduce that

$$\bar{u} = \bar{0} \text{ iff } (\forall\, \bar{x} \in E^n)\ \bar{x} \cdot \bar{u} = 0.$$

11. Prove that $\bar{x}$ and $\bar{y}$ are parallel iff

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n} = c \quad (c \in E^1),$$

where "$x_k/y_k = c$" is to be replaced by "$x_k = 0$" if $y_k = 0$.

12. Use induction on n to prove the *Lagrange identity* (valid in any field),

$$\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right) - \left(\sum_{k=1}^{n} x_k y_k\right)^2 = \sum_{1 \le i < k \le n} (x_i y_k - x_k y_i)^2.$$

Hence find a new proof of Theorem 4(c′).

13. Use Problem 7 and Theorem 4(c′) ("equality") to show that two nonzero vectors $\bar{x}$ and $\bar{y}$ in E^n are parallel iff $\cos\langle\bar{x}, \bar{y}\rangle = \pm 1$.

14. (i) Prove that $|\bar{x} + \bar{y}| = |\bar{x}| + |\bar{y}|$ iff $\bar{x} = t\bar{y}$ or $\bar{y} = t\bar{x}$ for some $t \ge 0$; equivalently, iff $\cos\langle\bar{x}, \bar{y}\rangle = 1$ (see Problem 7).

(ii) Find similar conditions for $|\bar{x} - \bar{y}| = |\bar{x}| + |\bar{y}|$.
[Hint: Look at the proof of Theorem 4(d′).]

§§4–6. Lines and Planes in E^n

I. To obtain a line in E^2 or E^3 passing through two points $\bar{a}$ and $\bar{b}$, we take the vector

$$\vec{u} = \overrightarrow{ab} = \bar{b} - \bar{a}$$

and, so to say, "stretch" it indefinitely in both directions, i.e., multiply $\vec{u}$ by all possible scalars $t \in E^1$. Then the set of all points $\bar{x}$ of the form

$$\bar{x} = \bar{a} + t\vec{u}$$

is the required line. It is natural to adopt this as a definition in E^n as well. Below, $\bar{a} \neq \bar{b}$.

Definition 1.

The *line* $\overline{ab}$ through the points $\bar{a}, \bar{b} \in E^n$ (also called the line through $\bar{a}$, in the direction of the vector $\vec{u} = \bar{b} - \bar{a}$) is the set of all points $\bar{x} \in E^n$ of the form

$$\bar{x} = \bar{a} + t\vec{u} = \bar{a} + t(\bar{b} - \bar{a}),$$

where t varies over E^1. We call t a variable real *parameter* and $\vec{u}$ a *direction vector* for $\overline{ab}$. Thus

$$\text{Line } \overline{ab} = \{\bar{x} \in E^n \mid \bar{x} = \bar{a} + t\vec{u} \text{ for some } t \in E^1\}, \quad \vec{u} = \bar{b} - \bar{a} \neq \bar{0}. \quad (1)$$

The formula

$$\bar{x} = \bar{a} + t\vec{u}, \text{ or } \bar{x} = \bar{a} + t(\bar{b} - \bar{a}),$$

is called the *parametric equation* of the line. (We briefly say "the line $\bar{x} = \bar{a} + t\vec{u}$.") It is equivalent to n simultaneous equations in terms of *coordinates*, namely,

$$x_k = a_k + tu_k = a_k + t(b_k - a_k), \quad k = 1, 2, \ldots, n. \tag{2}$$

Note 1. As the vector $\vec{u}$ is anyway being multiplied by all real numbers t, the line (as a set of points) will not change if $\vec{u}$ is replaced by some $c\vec{u}$ ($c \in E^1$, $c \neq 0$). In particular, taking $c = 1/|\vec{u}|$, we may replace $\vec{u}$ by $\vec{u}/|\vec{u}|$, *a unit vector*. We may as well assume that $\vec{u}$ is a unit vector itself.

If we let t vary not over all of E^1 but only over some interval in E^1, we obtain what is called a *line segment*.[1] In particular, we define the *open line segment* $L(\bar{a}, \bar{b})$, the *closed line segment* $L[\bar{a}, \bar{b}]$, the *half-open line segment* $L(\bar{a}, \bar{b}]$, and the *half-closed line segment* $L[\bar{a}, \bar{b})$, as we did for E^1.

Definition 2.

Given $\vec{u} = \bar{b} - \bar{a}$, we set

(i) $L(\bar{a}, \bar{b}) = \{\bar{a} + t\vec{u} \mid 0 < t < 1\}$;[2] (ii) $L[\bar{a}, \bar{b}] = \{\bar{a} + t\vec{u} \mid 0 \le t \le 1\}$;

(iii) $L(\bar{a}, \bar{b}] = \{\bar{a} + t\vec{u} \mid 0 < t \le 1\}$; (iv) $L[\bar{a}, \bar{b}) = \{\bar{a} + t\vec{u} \mid 0 \le t < 1\}$;

In all cases, $\bar{a}$ and $\bar{b}$ are called the *endpoints* of the segment; $\rho(\bar{a}, \bar{b}) = |\bar{b} - \bar{a}|$ is its *length*; and $\frac{1}{2}(\bar{a} + \bar{b})$ is its *midpoint*.

Note that in E^1, line segments simply become *intervals*, (a, b), $[a, b]$, etc.

II. To describe a *plane* in E^3, we fix one of its points, $\bar{a}$, and a vector $\vec{u} = \overrightarrow{ab}$ perpendicular to the plane (imagine a vertical pencil standing at $\bar{a}$ on the horizontal plane of the table). Then a point $\bar{x}$ lies on the plane iff $\vec{u} \perp \overrightarrow{ax}$. It is natural to accept this as a definition in E^n as well.

Definition 3.

Given a point $\bar{a} \in E^n$ and a vector $\vec{u} \neq \vec{0}$, we define the *plane* (also called *hyperplane* if $n > 3$) *through* $\bar{a}$*, orthogonal to* $\vec{u}$, to be the set of all $\bar{x} \in E^n$ such that $\vec{u} \perp \overrightarrow{ax}$, i.e., $\vec{u} \cdot (\bar{x} - \bar{a}) = 0$, or, in terms of components,

$$\sum_{k=1}^{n} u_k(x_k - a_k) = 0, \text{ where } \vec{u} \neq \vec{0} \text{ (i.e., not all values } u_k \text{ are 0).} \tag{3}$$

[1] We reserve the name "interval" for other kinds of sets (cf. §7).

[2] This is an abbreviation for "$\{\bar{x} \in E^n \mid \bar{x} = \bar{a} + t\vec{u}$ for some $t \in E^1$, $0 < t < 1\}$."

We briefly say

$$\text{“}\textit{the plane } \vec{u}\cdot(\bar{x}-\bar{a})=0\text{” or “}\textit{the plane } \sum_{k=1}^{n} u_k(x_k-a_k)=0\text{”}$$

(this being the *equation* of the plane). Removing brackets in (3), we have

$$u_1x_2+u_2x_2+\cdots+u_nx_n=c,\ \text{ or }\ \vec{u}\cdot\bar{x}=c,\ \text{ where } c=\sum_{k-1}^{n} u_k a_k,\ \ \vec{u}\neq\vec{0}. \quad (4)$$

An equation of this form is said to be *linear* in $x_1, x_2, \ldots, x_n$.

Theorem 1. *A set $A \subseteq E^n$ is a plane (hyperplane) iff A is exactly the set of all $\bar{x} \in E^n$ satisfying (4) for some fixed $c \in E^1$ and $\vec{u} = (u_1, \ldots, u_n) \neq \bar{0}$.*

Proof. Indeed, as we saw above, each plane has an equation of the form (4). Conversely, any equation of that form (with, say, $u_1 \neq 0$) can be written as

$$u_1\left(x_1-\frac{c}{u_1}\right)+u_2x_2+u_3x_3+\cdots+u_nx_n=0.$$

Then, setting $a_1 = c/u_1$ and $a_k = 0$ for $k \geq 2$, we transform it into (3), which is, by definition, the equation of a plane through $\bar{a} = (c/u_1, 0, \ldots, 0)$, orthogonal to $\vec{u} = (u_1, \ldots, u_n)$. □

Thus, briefly, planes are exactly all sets with *linear* equations (4). In this connection, equation (4) is called the *general equation* of a plane. The vector $\vec{u}$ is said to be *normal* to the plane. Clearly, if both sides of (4) are multiplied by a nonzero scalar q, one obtains an *equivalent* equation (representing *the same* set). Thus we may replace u_k by qu_k, i.e., $\vec{u}$ by $q\vec{u}$, without affecting the plane. In particular, we may replace $\vec{u}$ by the *unit vector* $\vec{u}/|\vec{u}|$, as in lines (this is called the *normalization* of the equation). Thus

$$\frac{\vec{u}}{|\vec{u}|}\cdot(\bar{x}-\bar{a})=0 \quad (5)$$

and

$$\bar{x}=\bar{a}+t\frac{\vec{u}}{|\vec{u}|} \quad (6)$$

are the *normalized* (or *normal*) equations of the plane (3) and line (1), respectively.

Note 2. The equation $x_k = c$ (for a fixed k) represents a *plane orthogonal to the basic unit vector $\vec{e}_k$* or, as we shall say, to *the kth axis*. The equation results from (4) if we take $\vec{u} = \vec{e}_k$ so that $u_k = 1$, while $u_i = 0$ $(i \neq k)$. For example, $x_1 = c$ is the equation of a plane orthogonal to $\vec{e}_1$; it consists of all $\bar{x} \in E^n$, with $x_1 = c$ (while the other coordinates of $\bar{x}$ are arbitrary). In E^2, it is a line. In E^1, it consists of c alone.

Two planes (respectively, two lines) are said to be *perpendicular* to each other iff their normal vectors (respectively, direction vectors) are orthogonal; similarly for parallelism. A plane $\vec{u} \cdot \bar{x} = c$ is said to be *perpendicular to a line* $\bar{x} = \bar{a} + t\vec{v}$ iff $\vec{u} \parallel \vec{v}$; the line and the plane are *parallel* iff $\vec{u} \perp \vec{v}$.

Note 3. When normalizing, as in (5) or (6), we actually have *two* choices of a unit vector, namely, $\pm\vec{u}/|\vec{u}|$. If one of them is *prescribed*, we speak of a *directed* plane (respectively, line).

Examples.

(a) Let $\bar{a} = (0, -1, 2)$, $\bar{b} = (1, 1, 1)$, and $\bar{c} = (3, 1, -1)$ in E^3. Then the line $\overline{ab}$ has the parametric equation $\bar{x} = \bar{a} + t(\bar{b} - \bar{a})$ or, in coordinates, writing x, y, z for x_1, x_2, x_3,

$$x = 0 + t(1 - 0) = t, \; y = -1 + 2t, \; z = 2 - t.$$

This may be rewritten

$$t = \frac{x}{1} = \frac{y+1}{2} = \frac{z-2}{-1},$$

where $\vec{u} = (1, 2, -1)$ is the direction vector (composed of the denominators). Normalizing and dropping t, we have

$$\frac{x}{1/\sqrt{6}} = \frac{y+1}{2/\sqrt{6}} = \frac{z-2}{-1/\sqrt{6}}$$

(the so-called *symmetric form* of the normal equations).

Similarly, for the line $\overline{bc}$, we obtain

$$t = \frac{x-1}{2} = \frac{y-1}{0} = \frac{z-1}{-2},$$

where "$t = (y-1)/0$" stands for "$y - 1 = 0$." (It is customary to use this notation.)

(b) Let $\bar{a} = (1, -2, 0, 3)$ and $\vec{u} = (1, 1, 1, 1)$ in E^4. Then the plane normal to $\vec{u}$ through $\bar{a}$ has the equation $(\bar{x} - \bar{a}) \cdot \vec{u} = 0$, or

$$(x_1 - 1) \cdot 1 + (x_2 + 2) \cdot 1 + (x_3 - 0) \cdot 1 + (x_4 - 3) \cdot 1 = 0,$$

or $x_1 + x_2 + x_3 + x_4 = 2$. Observe that, by formula (4), the coefficients of x_1, x_2, x_3, x_4 are the *components of the normal vector* $\vec{u}$ (here $(1, 1, 1, 1)$).

Now define a map $f\colon E^4 \to E^1$ setting $f(\bar{x}) = x_1 + x_2 + x_3 + x_4$ (the left-hand side of the equation). This map is called the *linear functional* corresponding to the given plane. (For another approach, see Problems 4–6 below.)

(c) The equation $x+3y-2z=1$ represents a plane in E^3, with $\vec{u}=(1,\,3,\,-2)$. The point $\bar{a}=(1,\,0,\,0)$ lies on the plane (why?), so the plane equation may be written $(\bar{x}-\bar{a})\cdot\vec{u}=0$ or $\bar{x}\cdot\vec{u}=1$, where $\bar{x}=(x,\,y,\,z)$ and $\bar{a}$ and $\vec{u}$ are as above.

Problems on Lines and Planes in E^n

1. Let $\bar{a}=(-1,\,2,\,0,\,-7)$, $\bar{b}=(0,\,0,\,-1,\,2)$, and $\bar{c}=(2,\,4,\,-3,\,-3)$ be points in E^4. Find the symmetric normal equations (see Example (a)) of the lines $\overline{ab}$, $\overline{bc}$, and $\overline{ca}$. Are any two of the lines perpendicular? Parallel? On the line $\overline{ab}$, find some points inside $L(\bar{a},\,\bar{b})$ and some outside $L[\bar{a},\,\bar{b}]$. Also, find the symmetric equations of the line through $\bar{c}$ that is

(i) parallel to $\overline{ab}$; (ii) perpendicular to $\overline{ab}$.

2. With $\bar{a}$ and $\bar{b}$ as in Problem 1, find the equations of the two planes that trisect, and are perpendicular to, the line segment $L[\bar{a},\,\bar{b}]$.

3. Given a line $\bar{x}=\bar{a}+t\vec{u}$ $(\vec{u}=\bar{b}-\bar{a}\neq\vec{0})$ in E^n, define $f\colon E^1\to E^n$ by

$$f(t)=\bar{a}+t\vec{u}\text{ for }t\in E^1.$$

Show that $L[\bar{a},\,\bar{b}]$ is exactly the f-image of the interval $[0,\,1]$ in E^1, with $f(0)=a$ and $f(1)=b$, while $f[E^1]$ is the entire line. Also show that f is one to one.

[Hint: $t\neq t'$ implies $|f(t)-f(t')|\neq 0$. Why?]

4. A map $f\colon E^n\to E^1$ is called a *linear functional* iff

$$(\forall\,\bar{x},\,\bar{y}\in E^n)\ (\forall\,a,\,b\in E^1)\quad f(a\bar{x}+b\bar{y})=af(\bar{x})+bf(\bar{y}).$$

Show by induction that f *preserves linear combinations*; that is,

$$f\left(\sum_{k=1}^{m}a_k\bar{x}_k\right)=\sum_{k=1}^{m}a_kf(\bar{x}_k)$$

for any $a_k\in E^1$ and $\bar{x}_k\in E^n$.

5. From Problem 4 prove that a map $f\colon E^n\to E^1$ is a linear functional iff there is $\vec{u}\in E^n$ such that

$$(\forall\,\bar{x}\in E^n)\quad f(\bar{x})=\vec{u}\cdot\bar{x}\quad(\text{“}\textit{representation theorem}\text{”}).$$

[Hint: If f is a linear functional, write each $\bar{x}\in E^n$ as $\bar{x}=\sum_{k=1}^{n}x_k\bar{e}_k$ (§§1–3, Theorem 2). Then

$$f(\bar{x})=f\left(\sum_{k=1}^{m}x_k\bar{e}_k\right)=\sum_{k=1}^{n}x_kf(\bar{e}_k).$$

Setting $u_k=f(\bar{e}_k)\in E^1$ and $\vec{u}=(u_1,\,\ldots,\,u_n)$, obtain $f(\bar{x})=\vec{u}\cdot\bar{x}$, as required. For the converse, use Theorem 3 in §§1–3.]

6. Prove that a set $A \subseteq E^n$ is a plane iff there is a linear functional f (Problem 4), *not identically zero*, and some $c \in E^1$ such that

$$A = \{\bar{x} \in E^n \mid f(\bar{x}) = c\}.$$

(This could serve as a *definition* of planes in E^n.)
[Hint: A is a plane iff $A = \{\bar{x} \mid \vec{u} \cdot \bar{x} = c\}$. Put $f(\bar{x}) = \vec{u} \cdot \bar{x}$ and use Problem 5. Show that $f \not\equiv 0$ iff $\vec{u} \neq \vec{0}$ by Problem 10 of §§1–3.]

7. Prove that the perpendicular distance of a point $\bar{p}$ to a plane $\vec{u} \cdot \bar{x} = c$ in E^n is

$$\rho(\bar{p}, \bar{x}_0) = \frac{|\vec{u} \cdot \bar{p} - c|}{|\vec{u}|}.$$

($\bar{x}_0$ is the *orthogonal projection* of $\bar{p}$, i.e., the point on the plane such that $\overrightarrow{px_0} \parallel \vec{u}$.)
[Hint: Put $\vec{v} = \vec{u}/|\vec{u}|$. Consider the line $\bar{x} = \bar{p} + t\vec{v}$. Find t for which $\bar{p} + t\vec{v}$ lies on both the line and plane. Find $|t|$.]

8. A *globe* (solid sphere) in E^n, with center $\bar{p}$ and radius $\varepsilon > 0$, is the set $\{\bar{x} \mid \rho(\bar{x}, \bar{p}) < \varepsilon\}$, denoted $G_{\bar{p}}(\varepsilon)$. Prove that if $\bar{a}, \bar{b} \in G_{\bar{p}}(\varepsilon)$, then also $L[\bar{a}, \bar{b}] \subseteq G_{\bar{p}}(\varepsilon)$. Disprove it for the *sphere* $S_{\bar{p}}(\varepsilon) = \{\bar{x} \mid \rho(\bar{x}, \bar{p}) = \varepsilon\}$.
[Hint: Take a line through $\bar{p}$.]

§7. Intervals in E^n

Consider the rectangle in E^2 shown in Figure 2. Its interior (without the perimeter) consists of all points $(x, y) \in E^2$ such that

$$a_1 < x < b_1 \text{ and } a_2 < y < b_2;$$

i.e.,

$$x \in (a_1, b_1) \text{ and } y \in (a_2, b_2).$$

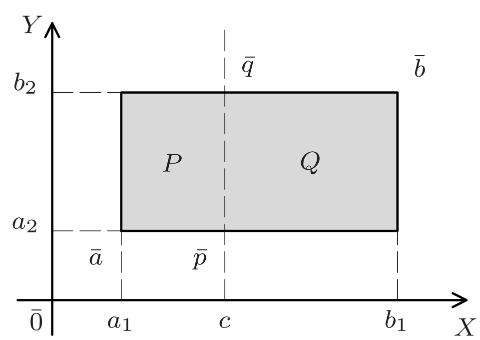

FIGURE 2

Thus it is the *Cartesian product of two line intervals*, (a_1, b_1) and (a_2, b_2). To include also all or some sides, we would have to replace open intervals by closed, half-closed, or half-open ones. Similarly, Cartesian products of three line intervals yield rectangular parallelepipeds in E^3. We call such sets in E^n *intervals*.

Definitions.

1. By an *interval* in E^n we mean the Cartesian product of any n intervals in E^1 (some may be open, some closed or half-open, etc.).

2. In particular, given

$$\bar{a} = (a_1, \ldots, a_n) \text{ and } \bar{b} = (b_1, \ldots, b_n)$$

with

$$a_k \leq b_k, \quad k = 1, 2, \ldots, n,$$

we define the *open interval* $(\bar{a}, \bar{b})$, the *closed interval* $[\bar{a}, \bar{b}]$, the *half-open interval* $(\bar{a}, \bar{b}]$, and the *half-closed interval* $[\bar{a}, \bar{b})$ as follows:

$$\begin{aligned}
(\bar{a}, \bar{b}) &= \{\bar{x} \mid a_k < x_k < b_k,\ k = 1, 2, \ldots, n\} \\
&= (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n); \\
[\bar{a}, \bar{b}] &= \{\bar{x} \mid a_k \leq x_k \leq b_k,\ k = 1, 2, \ldots, n\} \\
&= [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]; \\
(\bar{a}, \bar{b}] &= \{\bar{x} \mid a_k < x_k \leq b_k,\ k = 1, 2, \ldots, n\} \\
&= (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_n, b_n]; \\
[a, b) &= \{\bar{x} \mid a_k \leq x_k < b_k,\ k = 1, 2, \ldots, n\} \\
&= [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n).
\end{aligned}$$

In all cases, $\bar{a}$ and $\bar{b}$ are called the *endpoints* of the interval. Their distance

$$\rho(\bar{a}, \bar{b}) = |\bar{b} - \bar{a}|$$

is called its *diagonal*. The n differences

$$b_k - a_k = \ell_k \quad (k = 1, \ldots, n)$$

are called its n *edge-lengths*. Their product

$$\prod_{k=1}^{n} \ell_k = \prod_{k=1}^{n} (b_k - a_k)$$

is called the *volume* of the interval (in E^2 it is its *area*, in E^1 its *length*). The point

$$\bar{c} = \frac{1}{2}(\bar{a} + \bar{b})$$

is called its *center* or *midpoint*. The set difference

$$[\bar{a}, \bar{b}] - (\bar{a}, \bar{b})$$

is called the *boundary* of any interval with endpoints $\bar{a}$ and $\bar{b}$; it consists of $2n$ "faces" defined in a natural manner. (How?)

We often denote intervals by single letters, e.g., $A = (\bar{a}, \bar{b})$, and write dA for "diagonal of A" and vA or $\operatorname{vol} A$ for "volume of A." If all edge-lengths $b_k - a_k$

are *equal*, A is called a *cube* (in E^2, a *square*). The interval A is said to be *degenerate* iff $b_k = a_k$ for some k, in which case, clearly,

$$\operatorname{vol} A = \prod_{k=1}^{n} (b_k - a_k) = 0.$$

Note 1. We have $\bar{x} \in (\bar{a}, \bar{b})$ iff the inequalities $a_k < x_k < b_k$ hold *simultaneously* for all k. This is impossible if $a_k = b_k$ for some k; similarly for the inequalities $a_k < x_k \le b_k$ or $a_k \le x_k < b_k$. Thus *a degenerate interval is empty*, unless it is closed (in which case it contains $\bar{a}$ and $\bar{b}$ at least).

Note 2. In any interval A,

$$dA = \rho(\bar{a}, \bar{b}) = \sqrt{\sum_{k=1}^{n} (b_k - a_k)^2} = \sqrt{\sum_{k=1}^{n} \ell_k^2}\,.$$

In E^2, we can split an interval A into two subintervals P and Q by drawing a line (see Figure 2). In E^3, this is done by a *plane* orthogonal to one of the axes of the form $x_k = c$ (see §§4–6, Note 2), with $a_k < c < b_k$. In particular, if $c = \frac{1}{2}(a_k + b_k)$, the plane *bisects* the kth edge of A; and so the kth edge-length of P (and Q) equals $\frac{1}{2}\ell_k = \frac{1}{2}(b_k - a_k)$. If A is closed, so is P or Q, *depending on our choice*. (We may include the "partition" $x_k = c$ in P or Q.)[1]

Now, successively draw n planes $x_k = c_k$, $c_k = \frac{1}{2}(a_k + b_k)$, $k = 1, 2, \ldots, n$. The first plane bisects ℓ_j leaving the other edges of A unchanged. The resulting two subintervals P and Q then are cut by the plane $x_2 = c_2$, bisecting the second edge *in each of them*. Thus we get *four* subintervals (see Figure 3 for E^2). Each successive plane *doubles* the number of subintervals. After n steps, we thus obtain 2^n disjoint intervals, with *all* edges ℓ_k bisected. Thus by Note 2, the diagonal of each of them is

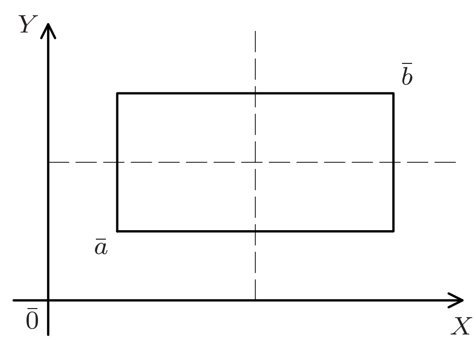

FIGURE 3

$$\sqrt{\sum_{k=1}^{n} \left(\frac{1}{2}\ell_k\right)^2} = \frac{1}{2}\sqrt{\sum_{k=1}^{n} \ell_k^2} = \frac{1}{2} dA.$$

Note 3. If A is *closed* then, as noted above, we can make any one (but *only one*) of the 2^n subintervals *closed* by properly manipulating each step.

The proof of the following simple corollaries is left to the reader.

[1] We have either $P = \{\bar{x} \in A \mid x_k \le c\}$ and $Q = \{\bar{x} \in A \mid x_k > c\}$, or $P = \{\bar{x} \in A \mid x_k < c\}$ and $Q = \{\bar{x} \in A \mid x_k \ge c\}$.

Corollary 1. *No distance between two points of an interval A exceeds dA, its diagonal. That is,* $(\forall\, \bar{x}, \bar{y} \in A)\ \rho(\bar{x}, \bar{y}) \leq dA$.

Corollary 2. *If an interval A contains $\bar{p}$ and $\bar{q}$, then also $L[\bar{p}, \bar{q}] \subseteq A$.*

Corollary 3. *Every nondegenerate interval in E^n contains rational points, i.e., points whose coordinates are all rational.*

(Hint: Use the density of rationals in E^1 for each coordinate separately.)

Problems on Intervals in E^n

(Here A and B denote *intervals*.)

1. Prove Corollaries 1–3.

2. Prove that if $A \subseteq B$, then $dA \leq dB$ and $vA \leq vB$.

3. Give an appropriate definition of a "face" and a "vertex" of A.

4. Find the edge-lengths of $A = (\bar{a}, \bar{b})$ in E^4 if

$$\bar{a} = (1, -2, 4, 0) \text{ and } \bar{b} = (2, 0, 5, 3).$$

Is A a cube? Find some *rational* points in it. Find dA and vA.

5. Show that the sets P and Q as defined in footnote 1 are *intervals*, indeed. In particular, they can be made half-open (half-closed) if A is half-open (half-closed).
[Hint: Let $A = (\bar{a}, \bar{b}]$,

$$P = \{\bar{x} \in A \mid x_k \leq c\}, \text{ and } Q = \{\bar{x} \in A \mid x_k > c\}.$$

To fix ideas, let $k = 1$, i.e., cut the *first* edge. Then let

$$\bar{p} = (c, a_2, \ldots, a_n) \text{ and } \bar{q} = (c, b_2, \ldots, b_n) \text{ (see Figure 2)},$$

and verify that $P = (\bar{a}, \bar{q}]$ and $Q = (\bar{p}, \bar{b}]$. Give a proof.]

6. In Problem 5, assume that A is *closed*, and make Q closed. (Prove it!)

7. In Problem 5 show that (with k fixed) the kth edge-lengths of P and Q equal $c - a_k$ and $b_k - c$, respectively, while for $i \neq k$ the edge-length ℓ_i is the same in A, P, and Q, namely, $\ell_i = b_i - a_i$.
[Hint: If $k = 1$, define $\bar{p}$ and $\bar{q}$ as in Problem 5.]

8. Prove that if an interval A is split into subintervals P and Q ($P \cap Q = \emptyset$), then $vA = vP + vQ$.
[Hint: Use Problem 7 to compute vA, vP, and vQ. Add up.]
Give an example. (Take A as in Problem 4 and split it by the plane $x_4 = 1$.)

***9.** Prove the *additivity* of the volume of intervals, namely, *if A is subdivided, in any manner, into m mutually disjoint subintervals $A_1, A_2, \ldots, A_m$*

in E^n, *then*

$$vA = \sum_{i=1}^{m} vA_i.$$

(This is true also if some A_i contain *common* faces).

[Proof outline: For $m = 2$, use Problem 8.

Then by induction, suppose additivity holds for any number of intervals *smaller* than a certain m ($m > 1$). Now let

$$A = \bigcup_{i=1}^{m} A_i \quad (A_i \text{ disjoint}).$$

One of the A_i (say, $A_1 = [\bar{a}, \bar{p}]$) must have some edge-length *smaller* than the corresponding edge-length of A (say, ℓ_1). Now cut all of A into $P = [\bar{a}, \bar{d}]$ and $Q = A - P$ (Figure 4) by the plane $x_1 = c$ ($c = p_1$) *so that* $A_1 \subseteq P$ *while* $A_2 \subseteq Q$. For simplicity, assume that the plane cuts *each* A_i into two subintervals A_i' and A_i''. (One of them may be empty.)

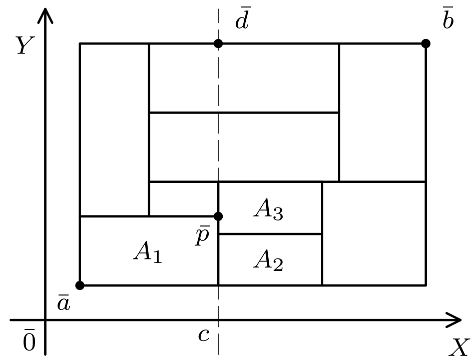

Figure 4

Then

$$P = \bigcup_{i=1}^{m} A_i' \text{ and } Q = \bigcup_{i=1}^{m} A_i''.$$

Actually, however, P and Q are split into *fewer* than m (nonempty) intervals since $A_1'' = \emptyset = A_2'$ by construction. Thus, by our inductive assumption,

$$vP = \sum_{i=1}^{m} vA_i' \text{ and } vQ = \sum_{i=1}^{m} vA_i'',$$

where $vA_1'' = 0 = vA_2'$, and $vA_i = vA_i' + vA_i''$ by Problem 8. Complete the inductive proof by showing that

$$vA = vP + vQ = \sum_{i=1}^{m} vA_i.]$$

§8. Complex Numbers

With all the operations defined in §§1–3, E^n ($n > 1$) is not yet a field because of the lack of a vector multiplication satisfying the field axioms. We shall now define such a multiplication, but only for E^2. Thus E^2 will become a field, which we shall call the *complex field*, C.

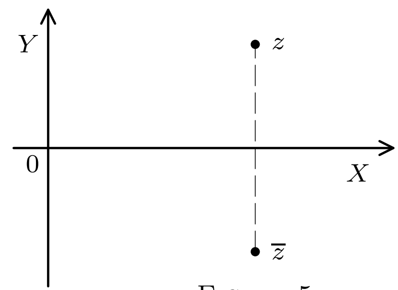

Figure 5

We make some changes in notation and terminology here. Points of E^2, when regarded as elements of C, will be called *complex numbers* (each being an ordered pair of real numbers). We denote them by single letters (preferably z) without a bar or an arrow. For example, $z = (x, y)$. We preferably write (x, y) for (x_1, x_2). If $z = (x, y)$, then x and y are called the *real* and *imaginary* parts of z, respectively,[1] and $\bar{z}$ denotes the complex number $(x, -y)$, called the *conjugate* of z (see Figure 5).

Complex numbers with vanishing imaginary part, $(x, 0)$, are called *real points* of C. For brevity, we simply write x for $(x, 0)$; for example, $2 = (2, 0)$. In particular, $1 = (1, 0) = \bar{\theta}_1$ is called the *real unit* in C. Points with vanishing real part, $(0, y)$, are called (*purely*) *imaginary* numbers. In particular, $\bar{\theta}_2 = (0, 1)$ is such a number; we shall now denote it by i and call it the *imaginary unit* in C. Apart from these peculiarities, all our former definitions of §§1–3 remain valid in $E^2 = C$. In particular, if $z = (x, y)$ and $z' = (x', y')$, we have

$$z \pm z' = (x, y) \pm (x', y') = (x \pm x', y \pm y'),$$
$$\rho(z, z') = \sqrt{(x - x')^2 + (y - y')^2}, \text{ and}$$
$$|z| = \sqrt{x^2 + y^2}.$$

All theorems of §§1–3 are valid.

We now define the new multiplication in C, which will make it a field.

Definition 1.

If $z = (x, y)$ and $z' = (x', y')$, then $zz' = (xx' - yy', xy' + yx')$.

Theorem 1. *$E^2 = C$ is a field, with zero element $0 = (0, 0)$ and unity $1 = (1, 0)$, under addition and multiplication as defined above.*

Proof. We only must show that *multiplication* obeys Axioms I–VI of the field axioms. Note that for addition, all is proved in Theorem 1 of §§1–3.

Axiom I (closure) is obvious from our definition, for if z and z' are in C, so is zz'.

To prove commutativity, take any complex numbers

$$z = (x, y) \text{ and } z' = (x', y')$$

[1] This terminology is solely traditional. Actually, there is nothing "imaginary" about $(0, y)$, no more than about $(x, 0)$, or (x, y).

and verify that $zz' = z'z$. Indeed, by definition,

$$zz' = (xx' - yy',\, xy' + yx') \text{ and } z'z = (x'x - y'y,\, x'y + y'x);$$

but the two expressions coincide by the commutative laws for *real* numbers. Associativity and distributivity are proved in a similar manner.

Next, we show that $1 = (1,\, 0)$ satisfies Axiom IV(b), i.e., that $1z = z$ for any complex number $z = (x,\, y)$. In fact, by definition, and by axioms for E^1,

$$1z = (1,\, 0)\,(x,\, y) = (1x - 0y,\, 1y + 0x) = (x - 0,\, y + 0) = (x,\, y) = z.$$

It remains to verify Axiom V(b), i.e., to show that each complex number $z = (x,\, y) \neq (0,\, 0)$ has an *inverse* z^{-1} such that $zz^{-1} = 1$. It turns out that the inverse is obtained by setting

$$z^{-1} = \left(\frac{x}{|z|^2},\, -\frac{y}{|z|^2}\right).$$

In fact, we then get

$$zz^{-1} = \left(\frac{x^2}{|z|^2} + \frac{y^2}{|z|^2},\, -\frac{xy}{|z|^2} + \frac{yx}{|z|^2}\right) = \left(\frac{x^2 + y^2}{|z|^2},\, 0\right) = (1,\, 0) = 1$$

since $x^2 + y^2 = |z|^2$, by definition. This completes the proof. □

Corollary 1. $i^2 = -1$; *i.e.*, $(0,\, 1)(0,\, 1) = (-1,\, 0)$.

Proof. By definition, $(0,\, 1)(0,\, 1) = (0 \cdot 0 - 1 \cdot 1,\, 0 \cdot 1 + 1 \cdot 0) = (-1,\, 0)$. □

Thus C *has an element* i *whose square is* -1, while E^1 has no such element, by Corollary 2 in Chapter 2, §§1–4. This is no contradiction since that corollary holds in *ordered* fields only. It only shows that C cannot be made an *ordered* field.

However, the "real points" in C form a subfield that *can* be ordered by setting

$$(x,\, 0) < (x',\, 0) \text{ iff } x < x' \text{ in } E^1.$$ [2]

Then *this subfield behaves exactly like* E^1.[3] Therefore, it is customary not to distinguish between "real points in C" and "real numbers," *identifying* $(x,\, 0)$ with x. With this convention, E^1 *simply is a subset* (*and a subfield*) *of* C. Henceforth, we shall simply say that "x is real" or "$x \in E^1$" instead of "$x = (x,\, 0)$ is a real point." We then obtain the following result.

Theorem 2. *Every* $z \in C$ *has a unique representation as*

$$z = x + yi,$$

[2] The proof is left as an exercise (Problem 1′ below).

[3] This can be made precise by using the notion of *isomorphism* (see *Basic Concepts of Mathematics*, Chapter 2, §14). We shall not go deeper into this topic here.

where x and y are real and $i = (0, 1)$. Specifically,

$$z = x + yi \text{ iff } z = (x, y).$$

Proof. By our conventions, $x = (x, 0)$ and $y = (y, 0)$, so

$$x + yi = (x, 0) + (y, 0)(0, 1).$$

Computing the right-hand expression from definitions, we have for any $x, y \in E^1$ that

$$x + yi = (x, 0) + (y \cdot 0 - 0 \cdot 1,\ y \cdot 1 + 0 \cdot 1) = (x, 0) + (0, y) = (x, y).$$

Thus $(x, y) = x + yi$ *for any* $x, y \in E^1$. In particular, if (x, y) is the given number $z \in C$ of the theorem, we obtain $z = (x, y) = x + yi$, as required.

To prove uniqueness, suppose that we also have

$$z = x' + y'i \text{ with } x' = (x', 0) \text{ and } y' = (y', 0).$$

Then, as shown above, $z = (x', y')$. Since also $z = (x, y)$, we have $(x, y) = (x', y')$, i.e., the two ordered pairs coincide, and so $x = x'$ and $y = y'$ after all. □

Geometrically, instead of Cartesian coordinates (x, y), we may also use *polar coordinates* r, θ, where

$$r = \sqrt{x^2 + y^2} = |z|$$

and θ is the (counterclockwise) rotation angle from the x-axis to the directed line $\overrightarrow{0z}$; see Figure 6. Clearly, z is uniquely determined by r and θ, but θ is *not* uniquely determined by z; indeed, the same point of E^2 results if θ is replaced by $\theta + 2n\pi$ $(n = 1, 2, \ldots)$. (If $z = 0$, then θ is not defined at all.) The values r and θ are called, respectively, the *modulus* and *argument* of $z = (x, y)$. By elementary trigonometry, $x = r\cos\theta$ and $y = r\sin\theta$. Substituting in $z = x + yi$, we obtain the following corollary.

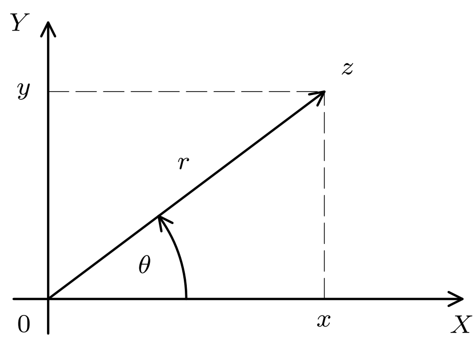

FIGURE 6

Corollary 2. $z = r(\cos\theta + i\sin\theta)$ (*trigonometric or polar form of* z).

Problems on Complex Numbers

1. Complete the proof of Theorem 1 (associativity, distributivity, etc.).

1′. Verify that the "real points" in C form an ordered field.

2. Prove that $z\bar{z} = |z|^2$. Deduce that $z^{-1} = \bar{z}/|z|^2$ if $z \neq 0$.[4]

3. Prove that
$$\overline{z + z'} = \bar{z} + \overline{z'} \text{ and } \overline{zz'} = \bar{z} \cdot \overline{z'}.$$
Hence show by induction that
$$\overline{z^n} = (\bar{z})^n,\ n = 1, 2, \ldots, \text{ and } \overline{\sum_{k=1}^{n} a_k z^k} = \sum_{k=1}^{n} \bar{a}_k \bar{z}^k.$$

4. Define
$$e^{\theta i} = \cos\theta + i\sin\theta.$$
Describe $e^{\theta i}$ geometrically. Is $|e^{\theta i}| = 1$?

5. Compute

(a) $\dfrac{1+2i}{3-i}$;

(b) $(1+2i)(3-i)$; and

(c) $\dfrac{x+1+i}{x+1-i}$, $x \in E^1$.

Do it in two ways: (i) using definitions only and the notation (x, y) for $x + yi$; and (ii) using all laws valid in a field.

6. Solve the equation $(2, -1)(x, y) = (3, 2)$ for x and y in E^1.

7. Let
$$\begin{aligned} z &= r(\cos\theta + i\sin\theta),\\ z' &= r'(\cos\theta' + i\sin\theta'), \text{ and}\\ z'' &= r''(\cos\theta'' + i\sin\theta'') \end{aligned}$$
as in Corollary 2. Prove that $z = z'z''$ if
$$r = |z| = r'r'', \text{ i.e., } |z'z''| = |z'|\,|z''|, \text{ and } \theta = \theta' + \theta''.$$
Discuss the following statement: To multiply z' by z'' means to rotate $\overrightarrow{0z'}$ counterclockwise by the angle θ'' and to multiply it by the scalar $r'' = |z''|$. Consider the cases $z'' = i$ and $z'' = -1$.

[Hint: Remove brackets in
$$r(\cos\theta + i\sin\theta) = r'(\cos\theta' + i\sin\theta') \cdot r''(\cos\theta'' + i\sin\theta'')$$
and apply the laws of trigonometry.]

8. By induction, extend Problem 7 to products of n complex numbers, and derive *de Moivre's formula*, namely, if $z = r(\cos\theta + i\sin\theta)$, then
$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

[4] Recall that $\bar{z}$ means "conjugate of z."

Use it to find, for $n = 1, 2, \ldots,$

$$\text{(a) } i^n; \qquad \text{(b) } (1+i)^n; \qquad \text{(c) } \frac{1}{(1+i)^n}.$$

9. From Problem 8, prove that for every complex number $z \neq 0$, there are exactly n complex numbers w such that

$$w^n = z;$$

they are called the *nth roots* of z.
[Hint: If

$$z = r(\cos\theta + i\sin\theta) \text{ and } w = r'(\cos\theta' + i\sin\theta'),$$

the equation $w^n = z$ yields, by Problem 8,

$$(r')^n = r \text{ and } n\theta' = \theta,$$

and conversely.

While this determines r' uniquely, θ may be replaced by $\theta + 2k\pi$ without affecting z. Thus

$$\theta' = \frac{\theta + 2k\pi}{n}, \quad k = 1, 2, \ldots.$$

Distinct points w result only from $k = 0, 1, \ldots, n-1$ (then they repeat cyclically). Thus n values of w are obtained.]

10. Use Problem 9 to find in C

(a) all cube roots of 1; (b) all fourth roots of 1.

Describe all nth roots of 1 geometrically.

*§9. Vector Spaces. The Space C^n. Euclidean Spaces

I. We shall now follow the pattern of E^n to obtain the general notion of a *vector space* (just as we generalized E^1 to define *fields*).

Let V be a set of arbitrary elements (not necessarily n-tuples), called "vectors" or "points," with a certain operation (call it "addition," $+$) somehow defined in V. Let F be any field (e.g., E^1 or C); its elements will be called *scalars*; its zero and unity will be denoted by 0 and 1, respectively. Suppose that yet another operation ("multiplication of scalars by vectors") has been defined that assigns to every scalar $c \in F$ and every vector $x \in V$ a certain vector, denoted cx or xc and called the *c-multiple of x*. Furthermore, suppose that *this multiplication and addition in V satisfy the nine laws specified in Theorem 1 of* §§1–3. That is, we have closure:

$$(\forall x, y \in V)\ (\forall c \in F) \quad x + y \in V \text{ and } cx \in V.$$

Vector addition is *commutative and associative*. There is a unique *zero-vector*, $\vec{0}$, such that

$$(\forall x \in V) \quad x + \vec{0} = x,$$

and each $x \in V$ has a unique *inverse*, $-x$, such that

$$x + (-x) = \vec{0}.$$

We have *distributivity*:

$$a(x + y) = ax + ay \text{ and } (a + b)x = ax + bx.$$

Finally, we have

$$1x = x$$

and

$$(ab)x = a(bx)$$

$(a,\, b \in F;\, x,\, y \in V)$.

In this case, V together with these two operations is called a *vector space* (or a *linear space*) *over the field* F; F is called its *scalar field*, and elements of F are called the *scalars* of V.

Examples.

(a) E^n is a vector space over E^1 (its scalar field).

(a′) R^n, the set of all *rational* points of E^n (i.e., points with rational coordinates) is a vector space *over* R, the rationals in E^1. (Note that we could take R as a scalar field for *all* of E^n; this would yield *another* vector space, E^n *over* R, not to be confused with E^n *over* E^1, i.e., the ordinary E^n.)

(b) Let F be any field, and let F^n be the set of all ordered n-tuples of elements of F, with sums and scalar multiples defined as in E^n (with F playing the role of E^1). Then F^n is a vector space over F (proof as in Theorem 1 of §§1–3).

(c) *Each field F is a vector space* (over itself) under the addition and multiplication defined in F. Verify!

(d) Let V be a vector space over a field F, and let W be the set of all possible mappings

$$f : A \to V$$

from some arbitrary set $A \neq \emptyset$ into V. Define the sum $f + g$ of two such maps by setting

$$(f + g)(x) = f(x) + g(x) \text{ for all } x \in A.$$[1]

[1] Here "$f+g$" must be treated as *one* letter (function symbol); "$(f+g)(x)$" means "$h(x)$," where $h = f + g$; similarly for such symbols as af, etc.

Similarly, given $a \in F$ and $f \in W$, define the map af by

$$(af)(x) = af(x).$$

Under these operations, W *is a vector space over the same field* F, with each map $f\colon A \to V$ treated as a single "vector" in W. (Verify!)

Vector spaces over E^1 (respectively, C) are called *real* (respectively, *complex*) linear spaces. Complex spaces can always be transformed into real ones by restricting their scalar field C to E^1 (treated as a subfield of C).

II. An important example of a complex linear space is C^n, the set of all ordered n-tuples

$$x = (x_1, \ldots, x_n)$$

of *complex* numbers x_k (now treated as *scalars*), with sums and scalar multiples defined as in E^n. In order to avoid confusion with conjugates of complex numbers, we shall not use the *bar* notation $\bar{x}$ for a vector in this section, writing simply x for it. Dot products in C^n are defined by

$$x \cdot y = \sum_{k=1}^{n} x_k \bar{y}_k,$$

where $\bar{y}_k$ is the conjugate of the complex number y_k (see §8), and hence a *scalar* in C. Note that $\bar{y}_k = y_k$ if $y_k \in E^1$. Thus, for vectors with *real* components,

$$x \cdot y = \sum_{k=1}^{n} x_k y_k,$$

as in E^n. The reader will easily verify (exactly as for E^n) that, for $x, y \in C^n$ and $a, b \in C$, we have the following properties:

(i) $x \cdot y \in C$; thus $x \cdot y$ is a *scalar*, not a vector.

(ii) $x \cdot x \in E^1$, and $x \cdot x \geq 0$; moreover, $x \cdot x = 0$ iff $x = \vec{0}$. (Thus *the dot product of a vector by itself is a real number* ≥ 0.)

(iii) $x \cdot y = \overline{y \cdot x}$ (= conjugate of $y \cdot x$). Commutativity *fails* in general.

(iv) $(ax) \cdot (by) = (a\bar{b})(x \cdot y)$. Hence (iv′) $(ax) \cdot y = a(x \cdot y) = x \cdot (\bar{a}y)$.

(v) $(x + y) \cdot z = x \cdot z + y \cdot z$ and (v′) $z \cdot (x + y) = z \cdot x + z \cdot y$.

Observe that (v′) follows from (v) by (iii). (Verify!)

III. Sometimes (but not always) dot products can also be defined in real or complex linear spaces other than E^n or C^n, in such a manner as to satisfy the laws (i)–(v), hence also (v′), listed above, with C replaced by E^1 if the space is real. If these laws hold, the space is called *Euclidean*. For example, E^n is a real Euclidean space and C^n is a complex one.

In every such space, we define *absolute values* of vectors by

$$|x| = \sqrt{x \cdot x}.$$

(This root exists in E^1 by formula (ii).) In particular, this applies to E^n and C^n. Then given any vectors x, y and a scalar a, we obtain as before the following properties:

(a′) $|x| \geq 0$; and $|x| = 0$ iff $x = \vec{0}$.

(b′) $|ax| = |a|\,|x|$.

(c′) *Triangle inequality*: $|x + y| \leq |x| + |y|$.

(d′) *Cauchy–Schwarz inequality*: $|x \cdot y| \leq |x|\,|y|$, and $|x \cdot y| = |x|\,|y|$ iff $x \parallel y$ (i.e., $x = ay$ or $y = ax$ for some scalar a).

We prove only (d′); the rest is proved as in Theorem 4 of §§1–3.

If $x \cdot y = 0$, all is trivial, so let $z = x \cdot y = rc \neq 0$, where $r = |x \cdot y|$ and c has modulus 1, and let $y' = cy$. For any (variable) $t \in E^1$, consider $|tx + y'|$. By definition and (v), (iii), and (iv),

$$\begin{aligned}|tx + y'|^2 &= (tx + y') \cdot (tx + y')\\ &= tx \cdot tx + y' \cdot tx + tx \cdot y' + y' \cdot y'\\ &= t^2(x \cdot x) + t(y' \cdot x) + t(x \cdot y') + (y' \cdot y')\end{aligned}$$

since $\bar{t} = t$. Now, since $c\bar{c} = 1$,

$$x \cdot y' = x \cdot (cy) = (\bar{c}x) \cdot y = \bar{c}rc = r = |x \cdot y|.$$

Similarly,

$$y' \cdot x = \overline{x \cdot y'} = \bar{r} = r = |x \cdot y|,\ x \cdot x = |x|^2, \text{ and } y' \cdot y' = y \cdot y = |y|^2.$$

Thus we obtain

$$(\forall\, t \in E^1) \quad |tx + cy|^2 = t^2|x|^2 + 2t|x \cdot y| + |y|^2. \tag{1}$$

Here $|x|^2$, $2|x \cdot y|$, and $|y|^2$ are fixed *real* numbers. We treat them as coefficients in t of the quadratic trinomial

$$f(t) = t^2|x|^2 + 2t|x \cdot y| + |y|^2.$$

Now if x and y are *not* parallel, then $cy \neq -tx$, and so

$$|tx + cy| = |tx + y'| \neq 0$$

for *any* $t \in E^1$. Thus by (1), the quadratic trinomial has no real roots; hence its discriminant,

$$4|x \cdot y|^2 - 4(|x|\,|y|)^2,$$

is negative, so that $|x \cdot y| < |x|\,|y|$.

If, however, $x \parallel y$, one easily obtains $|x \cdot y| = |x|\,|y|$, by (b′). (Verify.)

Thus $|x \cdot y| = |x|\,|y|$ or $|x \cdot y| < |x|\,|y|$ according to whether $x \parallel y$ or not. □

In any Euclidean space, we define *distances* by $\rho(x, y) = |x - y|$. *Planes*, *lines*, *and line segments* are defined exactly as in E^n. Thus

$$\text{line } \overline{pq} = \{p + t(q - p) \mid t \in E^1\} \text{ (in real and complex spaces alike).}$$

Problems on Linear Spaces

1. Prove that F^n in Example (b) is a vector space, i.e., that it satisfies all laws stated in Theorem 1 in §§1–3; similarly for W in Example (d).

2. Verify that dot products in C^n obey the laws (i)–(v′). Which of these laws would fail if these products were defined by

$$x \cdot y = \sum_{k=1}^{n} x_k y_k \text{ instead of } x \cdot y = \sum_{k=1}^{n} x_k \bar{y}_k?$$

How would this affect the properties of absolute values given in (a′)–(d′)?

3. Complete the proof of formulas (a′)–(d′) for Euclidean spaces. What change would result if property (ii) of dot products were restated as

$$\text{“}x \cdot x \geq 0 \text{ and } \vec{0} \cdot \vec{0} = 0\text{”?}$$

4. Define orthogonality, parallelism and *angles* in a general Euclidean space following the pattern of §§1–3 (text and Problem 7 there). Show that $u = \vec{0}$ iff u is orthogonal to *all* vectors of the space.

5. Define the basic unit vectors e_k in C^n exactly as in E^n, and prove Theorem 2 in §§1–3 for C^n (replacing E^1 by C). Also, do Problem 5(a) of §§1–3 for C^n.

6. Define hyperplanes in C^n as in Definition 3 of §§4–6, and prove Theorem 1 stated there, for C^n. Do also Problems 4–6 there for C^n (replacing E^1 by C) and Problem 4 there for vector spaces in general (replacing E^1 by the scalar field F).

7. Do Problem 3 of §§4–6 for general Euclidean spaces (real or complex). Note: Do *not* replace E^1 by C in the definition of a line and a line segment.

8. A finite set of vectors $B = \{x_1, \ldots, x_m\}$ in a linear space V over F is said to be *independent* iff

$$(\forall\, a_1, a_2, \ldots, a_m \in F) \quad \left(\sum_{i=1}^{m} a_i x_i = \vec{0} \Longrightarrow a_1 = a_2 = \cdots = a_m = 0\right).$$

Prove that if B is independent, then

(i) $\vec{0} \notin B$;

(ii) each subset of B is independent ($\emptyset$ counts as independent); and

(iii) if for some scalars a_i, $b_i \in F$,

$$\sum_{i=1}^{m} a_i x_i = \sum_{i=1}^{m} b_i x_i,$$

then $a_i = b_i$, $i = 1, 2, \ldots, m$.

9. Let V be a vector space over F and let $A \subseteq V$. By the *span of* A in V, denoted span(A), is meant the set of all "linear combinations" of vectors from A, i.e., all vectors of the form

$$\sum_{i=1}^{m} a_i x_i, \quad a_i \in F,\ x_i \in A,\ m \in N.$$[2]

Show that span(A) is itself a vector space $V' \subseteq V$ (a *subspace* of V) over the same field F, with the operations defined in V. (We say that A *spans* V'.) Show that in E^n and C^n, the basic unit vectors span the entire space.

*§10. Normed Linear Spaces

By a *normed linear space* (briefly *normed space*) is meant a real or complex vector space E in which every vector x is associated with a real number $|x|$, called its *absolute value* or *norm*, in such a manner that the properties (a′)–(c′) of §9 hold.[1] That is, for any vectors $x, y \in E$ and scalar a, we have

(i) $|x| \geq 0$;

(i′) $|x| = 0$ iff $x = \vec{0}$;

(ii) $|ax| = |a|\,|x|$; and

(iii) $|x + y| \leq |x| + |y|$ (triangle inequality).

Mathematically, the existence of absolute values in E amounts to that of a map (called a *norm map*) $x \to |x|$ on E, i.e., a map $\varphi\colon E \to E^1$, with function values $\varphi(x)$ written as $|x|$, satisfying the laws (i)–(iii) above. Often such a map can be chosen in *many* ways (not necessarily via dot products, which may not exist in E), thus giving rise to *different* norms on E. Sometimes we write $\|x\|$ for $|x|$ or use other similar symbols.

Note 1. From (iii), we also obtain $|x - y| \geq \big||x| - |y|\big|$ exactly as in E^n.

[2] If $A = \emptyset$, then span$(A) = \{\vec{0}\}$ by definition.

[1] Roughly, it is a vector space (*over* E^1 *or* C) in which "well-behaved" absolute values are defined, resembling those in E^n.

Examples.

(A) Each *Euclidean space* (§9), such as E^n or C^n, is a normed space, with norm defined by

$$|x| = \sqrt{x \cdot x}\,,$$

as follows from formulas (a′)–(c′) in §9. In E^n and C^n, one can also equivalently define

$$|x| = \sqrt{\sum_{k=1}^{n} |x_k|^2},$$

where $x = (x_1, \ldots, x_n)$. This is the so-called *standard* norm, usually presupposed in E^n (C^n).

(B) One can also define other, "*nonstandard*," norms on E^n and C^n. For example, fix some real $p \ge 1$ and put

$$|x|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}.$$

One can show that $|x|_p$ so defined satisfies (i)–(iii) and thus is a *norm* (see Problems 5–7 below).

(C) Let W be the set of all *bounded* maps

$$f\colon A \to E$$

from a set $A \neq \emptyset$ into a normed space E, i.e., such that

$$(\forall\, t \in A) \quad |f(t)| \le c$$

for some real constant $c > 0$ (dependent on f but not on t). Define $f + g$ and af as in Example (d) of §9 so that W becomes a vector space. Also, put

$$\|f\| = \sup_{t \in A} |f(t)|,$$

i.e., the supremum of all $|f(t)|$, with $t \in A$. Due to boundedness, this supremum exists *in* E^1, so $\|f\| \in E^1$.

It is easy to show that $\|f\|$ is a norm on W. For example, we verify (iii) as follows.

By definition, we have for $f, g \in W$ and $x \in A$,

$$\begin{aligned} |(f+g)(x)| &= |f(x) + g(x)| \\ &\le |f(x)| + |g(x)| \\ &\le \sup_{t \in A} |f(t)| + \sup_{t \in A} |g(t)| \\ &= \|f\| + \|g\|. \end{aligned} \tag{1}$$

(The first inequality is true because (iii) *holds in the normed space* E to which $f(x)$ and $g(x)$ belong.) By (1), $\|f\| + \|g\|$ is an upper bound of all expressions $|(f+g)(x)|$, $x \in A$. Thus

$$\|f\| + \|g\| \geq \sup_{x \in A} |(f+g)(x)| = \|f+g\|.$$

Note 2. Formula (1) also shows that the map $f+g$ is *bounded* and hence is a member of W. Quite similarly we see that $af \in W$ for any scalar a and $f \in W$. Thus we have the closure laws for W. The rest is easy.

In every normed (in particular, in each Euclidean) space E, we define *distances* by

$$\rho(x, y) = |x - y| \quad \text{for all } x, y \in E.$$

Such distances depend, of course, on the norm chosen for E; thus we call them *norm-induced* distances. In particular, using the *standard* norm in E^n and C^n (Example (A)), we have

$$\rho(x, y) = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2}.$$

Using the norm of Example (B), we get

$$\rho(x, y) = \left(\sum_{k=1}^{n} |x_k - y_k|^p\right)^{\frac{1}{p}}$$

instead. In the space W of Example (C), we have

$$\rho(f, g) = \|f - g\| = \sup_{x \in A} |f(x) - g(x)|.$$

Proceeding exactly as in the proof of Theorem 5 in §§1–3, we see that norm-induced distances *obey the three laws stated there*. (Verify!) Moreover, by definition,

$$\rho(x+u, y+u) = |(x+u) - (y+u)| = |x - y| = \rho(x, y).$$

Thus we have

$$\rho(x, y) = \rho(x+u, y+u) \text{ for norm-induced distances;} \tag{2}$$

i.e., *the distance* $\rho(x, y)$ *does not change if both* x *and* y *are "translated" by one and the same vector* u. We call such distances *translation-invariant*.

A more general theory of distances will be given in §§11ff.

Problems on Normed Linear Spaces

1. Show that distances in normed spaces obey the laws stated in Theorem 5 of §§1–3.

2. Complete the proof of assertions made in Example (C) and Note 2.

3. Define $|x| = x_1$ for $x = (x_1, \ldots, x_n)$ in C^n or E^n. Is this a norm? Which (if any) of the laws (i)–(iii) does it obey? How about formula (2)?

4. Do Problem 3 in §§4–6 for a general normed space E, with lines defined as in E^n (see also Problem 7 in §9). Also, show that *contracting* sequences of line segments in E are f-images of *contracting* sequences of intervals in E^1. Using this fact, deduce from Problem 11 in Chapter 2, §§8–9, an analogue for *line segments* in E, namely, if

$$L[a_n, b_n] \supseteq L[a_{n+1}, b_{n+1}], \quad n = 1, 2, \ldots,$$

then

$$\bigcap_{n=1}^{\infty} L[a_n, b_n] \neq \emptyset.$$

5. Take for granted the lemma that

$$a^{1/p} b^{1/q} \leq \frac{a}{p} + \frac{b}{q}$$

if $a, b, p, q \in E^1$ with $a, b \geq 0$ and $p, q > 0$, and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

(A proof will be suggested in Chapter 5, §6, Problem 11.) Use it to prove *Hölder's inequality*, namely, if $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}} \text{ for any } x_k, y_k \in C.$$

[Hint: Let

$$A = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \text{ and } B = \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}}.$$

If $A = 0$ or $B = 0$, then all x_k or all y_k vanish, and the required inequality is trivial. Thus assume $A \neq 0$ and $B \neq 0$. Then, setting

$$a = \frac{|x_k|^p}{A^p} \text{ and } b = \frac{|y_k|^q}{B^q}$$

in the lemma, obtain

$$\frac{|x_k y_k|}{AB} \leq \frac{|x_k|^p}{pA^p} + \frac{|y_k|^q}{qB^q}, \; k = 1, 2, \ldots, n.$$

Now add up these n inequalities, substitute the values of A and B, and simplify.]

6. Prove the *Minkowski inequality*,

$$\left(\sum_{k=1}^{n}|x_k+y_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{n}|x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n}|y_k|^p\right)^{\frac{1}{p}}$$

for any real $p \ge 1$ and $x_k,\ y_k \in C$.
[Hint: If $p = 1$, this follows by the triangle inequality in C. If $p > 1$, let

$$A = \sum_{k=1}^{n}|x_k+y_k|^p \neq 0.$$

(If $A = 0$, all is trivial.) Then verify (writing "$\sum$" for "$\sum_{k=1}^{n}$" for simplicity)

$$A = \sum |x_k+y_k||x_k+y_k|^{p-1} \le \sum |x_k||x_k+y_k|^{p-1} + \sum |y_k||x_k+y_k|^{p-1}$$

Now apply Hölder's inequality (Problem 5) to each of the last two sums, with $q = p/(p-1)$, so that $(p-1)q = p$ and $1/p = 1 - 1/q$. Thus obtain

$$A \le \left(\sum |x_k|^p\right)^{\frac{1}{p}}\left(\sum |x_k+y_k|^p\right)^{\frac{1}{q}} + \left(\sum |y_k|^p\right)^{\frac{1}{p}}\left(\sum |x_k+y_k|^p\right)^{\frac{1}{q}}.$$

Then divide by $A^{\frac{1}{q}} = (\sum |x_k+y_k|^p)^{\frac{1}{q}}$ and simplify.]

7. Show that Example (B) indeed yields a *norm* for C^n and E^n.
[Hint: For the triangle inequality, use Problem 6. The rest is easy.]

8. A sequence $\{x_m\}$ of vectors in a normed space E (e.g., in E^n or C^n) is said to be *bounded* iff

$$(\exists\, c \in E^1)\ (\forall\, m) \quad |x_m| < c,$$

i.e., iff $\sup_m |x_m|$ is *finite*.

Denote such sequences by single letters, $x = \{x_m\}$, $y = \{y_m\}$, etc., and define

$$x + y = \{x_m + y_m\}, \text{ and } ax = \{ax_m\} \text{ for any scalar } a.$$

Also let

$$|x| = \sup_m |x_m|.$$

Show that, with these definitions, the set M of all bounded infinite sequences in E becomes a normed space (in which every such sequence is to be treated as a single vector, and the scalar field is the same as that of E).

§11. Metric Spaces

I. In §§1–3, we defined *distances* $\rho(\bar{x}, \bar{y})$ for points $\bar{x}, \bar{y}$ in E^n using the formula

$$\rho(\bar{x}, \bar{y}) = \sqrt{\sum_{k=1}^{n}(x_k - y_k)^2} = |\bar{x} - \bar{y}|.$$

This actually amounts to defining a certain *function* ρ *of two variables* $\bar{x}, \bar{y} \in E^n$. We also showed that ρ obeys the three laws of Theorem 5 there. (We call them *metric* laws.)

Now, as will be seen, such functions ρ can also be defined in other sets, using quite different defining formulas. In other words, given any set $S \neq \emptyset$ of arbitrary elements, one can define in it, so to say, "fancy distances" $\rho(x, y)$ satisfying the same three laws. It turns out that it is not the particular formula used to define ρ but rather the preservation of the three laws that is most important for general theoretical purposes.

Thus we shall assume that a function ρ with the same three properties has been defined, *in some way or other*, for a set $S \neq \emptyset$, and propose to study the consequences of the three metric laws *alone, without assuming anything else*. (In particular, no operations other than ρ, or absolute values, or inequalities $<$, need be defined in S.) All results so obtained will, of course, apply to distances in E^n (since they obey the metric laws), but *they will also apply to other cases where the metric laws hold*.

The elements of S (though arbitrary) will be called "*points*," usually denoted by p, q, x, y, z (sometimes with bars, etc.); ρ is called a *metric* for S. We symbolize it by

$$\rho\colon S \times S \to E^1$$

since it is function defined *on* $S \times S$ (*pairs* of elements of S) into E^1. Thus we are led to the following definition.

Definition 1.

A *metric space* is a set $S \neq \emptyset$ together with a function

$$\rho\colon S \times S \to E^1$$

(called a *metric* for S) satisfying the *metric laws* (*axioms*):

For any x, y, and z in S, we have

(i) $\rho(x, y) \geq 0$, and (i') $\rho(x, y) = 0$ iff $x = y$;

(ii) $\rho(x, y) = \rho(y, x)$ (symmetry law); and

(iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ (triangle law).

Thus a metric space is a *pair* $(S,\, \rho)$, namely, a set S and a metric ρ for it. In general, one can define many different metrics

$$\rho,\, \rho',\, \rho'',\, \ldots$$

for the same S. The resulting spaces

$$(S,\, \rho),\, (S, \rho'),\, (S, \rho''),\, \ldots$$

then are regarded as *different*. However, if confusion is unlikely, we simply write S for $(S,\, \rho)$. We write "$p \in (S,\, \rho)$" for "$p \in S$ with metric ρ," and "$A \subseteq (S,\, \rho)$" for "$A \subseteq S$ in $(S,\, \rho)$."

Examples.

(1) In E^n, we always assume

$$\rho(\bar{x},\, \bar{y}) = |\bar{x} - \bar{y}| \quad \text{(the "\textit{standard metric}")}$$

unless stated otherwise.[1] By Theorem 5 in §§1–3, $(E^n,\, \rho)$ is a metric space.

(2) However, one can define for E^n many other "nonstandard" metrics. For example,

$$\rho'(\bar{x},\, \bar{y}) = \left(\sum_{k=1}^{n} |x_k - y_k|^p \right)^{1/p} \text{ for any real } p \geq 1$$

likewise satisfies the metric laws (a proof is suggested in §10, Problems 5–7); similarly for C^n.

(3) *Any* set $S \neq \emptyset$ can be "metrized" (i.e., endowed with a metric) by setting

$$\rho(x,\, y) = 1 \text{ if } x \neq y, \text{ and } \rho(x,\, x) = 0.$$

(Verify the metric laws!) This is the so-called *discrete metric*. The space $(S,\, \rho)$ so defined is called a *discrete space*.

(4) Distances ("mileages") on the surface of our planet are actually measured along circles fitting in the curvature of the globe (not straight lines). One can show that they obey the metric laws and thus define a (nonstandard) metric for $S =$ (surface of the globe).

(5) A mapping $f\colon A \to E^1$ is said to be *bounded* iff

$$(\exists\, K \in E^1)\ (\forall\, x \in A) \quad |f(x)| \leq K.$$

[1] Similarly in other *normed spaces* (§10), such as C^n. (A reader who has omitted the "starred" §10 will consider E^n only.)

For a fixed $A \neq \emptyset$, let W be the set of all such maps (each being treated as a single "point" of W). Metrize W by setting, for $f, g \in W$,

$$\rho(f, g) = \sup_{x \in A} |f(x) - g(x)|.$$

(Verify the metric laws; see a similar proof in §10.)

II. We now define "balls" in *any* metric space (S, ρ).

Definition 2.

Given $p \in (S, \rho)$ and a real $\varepsilon > 0$, we define the *open ball* or *globe* with *center* p and *radius* ε (briefly "ε-globe about p"), denoted

$$G_p \text{ or } G_p(\varepsilon) \text{ or } G(p; \varepsilon),$$

to be *the set of all* $x \in S$ *such that*

$$\rho(x, p) < \varepsilon.$$

Similarly, the *closed* ε*-globe about* p is

$$\overline{G}_p = \overline{G}_p(\varepsilon) = \{x \in S \mid \rho(x, p) \leq \varepsilon\}.$$

The ε*-sphere about* p is defined by

$$S_p(\varepsilon) = \{x \in S \mid \rho(x, p) = \varepsilon\}.$$

Note. An open globe in E^3 is an ordinary *solid sphere* (without its surface $S_p(\varepsilon)$), as known from geometry. In E^2, an open globe is a *disc* (the interior of a circle). *In* E^1*, the globe* $G_p(\varepsilon)$ *is simply the open interval*

$$(p - \varepsilon, p + \varepsilon),$$

while $\overline{G}_p(\varepsilon)$ is the *closed* interval

$$[p - \varepsilon, p + \varepsilon].$$

The shape of the globes and spheres *depends on the metric* ρ. It may become rather strange for various unusual metrics. For example, in the *discrete space* (Example (3)), any globe of radius < 1 consists of its center alone, while $G_p(2)$ contains the entire space. (Why?) See also Problems 1, 2, and 4.

III. Now take any nonempty set

$$A \subseteq (S, \rho).$$

The distances $\rho(x, y)$ in S are, of course, also defined for points of A (since $A \subseteq S$), and the metric laws remain valid in A. *Thus* A *is likewise a (smaller) metric space under the metric* ρ *"inherited" from* S; we only have to restrict the domain of ρ to $A \times A$ (pairs of points from A). The set A with this metric is called a *subspace* of S. We shall denote it by (A, ρ), using the same letter ρ, or simply by A. Note that A with some *other* metric ρ' is *not* called a subspace of (S, ρ).

By definition, points in (A, ρ) have the same distances as in (S, ρ). However, globes and spheres in (A, ρ) must consist of points from A only, with centers in A. Denoting such a globe by

$$G_p^*(\varepsilon) = \{x \in A \mid \rho(x, p) < \varepsilon\},$$

we see that it is obtained by *restricting* $G_p(\varepsilon)$ (*the corresponding globe in* S) *to points of* A, i.e., removing all points not in A. Thus

$$G_p^*(\varepsilon) = A \cap G_p(\varepsilon);$$

similarly for closed globes and spheres. $A \cap G_p(\varepsilon)$ is often called the *relativized* (to A) globe $G_p(\varepsilon)$. Note that $p \in G_p^*(\varepsilon)$ since $\rho(p, p) = 0 < \varepsilon$, and $p \in A$.

For example, let R be the subspace of E^1 consisting of rationals only. Then the relativized globe $G_p^*(\varepsilon)$ consists of all rationals in the interval

$$G_p(\varepsilon) = (p - \varepsilon, p + \varepsilon),$$

and it is assumed here that p is rational itself.

IV. A few remarks are due on the *extended real number system* E^* (see Chapter 2, §13). As we know, E^* consists of all reals and two additional elements, $\pm\infty$, with the convention that $-\infty < x < +\infty$ for all $x \in E^1$. The standard metric ρ does not apply to E^*. However, one can metrize E^* in various other ways. The most common metric ρ' is suggested in Problems 5 and 6 below. *Under that metric, globes turn out to be finite and infinite intervals in* E^*.

Instead of metrizing E^*, we may simply adopt the convention that intervals of the form

$$(a, +\infty] \text{ and } [-\infty, a), \ a \in E^1,$$

will be called "globes" about $+\infty$ and $-\infty$, respectively (without specifying any "radii"). Globes about *finite* points may remain as they are in E^1. This convention suffices for most purposes of limit theory. We shall use it often (as we did in Chapter 2, §13).

Problems on Metric Spaces

The "arrowed" problems should be noted for later work.

1. Show that E^2 becomes a metric space if distances $\rho(\bar{x}, \bar{y})$ are defined by

(a) $\rho(\bar{x}, \bar{y}) = |x_1 - y_1| + |x_2 - y_2|$ or

(b) $\rho(\bar{x}, \bar{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$. In each case, describe $G_{\bar{0}}(1)$ and $S_{\bar{0}}(1)$. Do the same for the subspace of points with *nonnegative* coordinates.

2. Prove the assertions made in the text about globes in a discrete space. Find an *empty sphere* in such a space. Can a sphere contain the entire space?

3. Show that ρ in Examples (3) and (5) obeys the metric axioms.

4. Let M be the set of all positive integers together with the "point" ∞. Metrize M by setting

$$\rho(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right|, \text{ with the convention that } \frac{1}{\infty} = 0.$$

Verify the metric axioms. Describe $G_\infty(\frac{1}{2})$, $S_\infty(\frac{1}{2})$, and $G_1(1)$.

⇒5. Metrize the extended real number system E^* by

$$\rho'(x, y) = |f(x) - f(y)|,$$

where the function

$$f\colon E^* \underset{\text{onto}}{\longrightarrow} [-1, 1]$$

is defined by

$$f(x) = \frac{x}{1 + |x|} \text{ if } x \text{ is finite, } f(-\infty) = -1, \text{ and } f(+\infty) = 1.$$

Compute $\rho'(0, +\infty)$, $\rho'(0, -\infty)$, $\rho'(-\infty, +\infty)$, $\rho'(0, 1)$, $\rho'(1, 2)$, and $\rho'(n, +\infty)$. Describe $G_0(1)$, $G_{+\infty}(1)$, and $G_{-\infty}(\frac{1}{2})$. Verify the metric axioms (also when infinities are involved).

⇒6. In Problem 5, show that the function f is one to one, *onto* $[-1, 1]$, and *increasing*; i.e.,

$$x < x' \text{ implies } f(x) < f(x') \text{ for } x, x' \in E^*.$$

Also show that the f-image of an interval $(a, b) \subseteq E^*$ is the interval $(f(a), f(b))$. Hence deduce that globes in E^* (with ρ' as in Problem 5) are *intervals* in E^* (possibly infinite).

[Hint: For a *finite* x, put

$$y = f(x) = \frac{x}{1 + |x|}.$$

Solving for x (separately in the cases $x \geq 0$ and $x < 0$), show that

$$(\forall\, y \in (-1, 1)) \quad x = f^{-1}(y) = \frac{y}{1 - |y|};$$

thus x is *uniquely* determined by y, i.e., f is one to one and *onto—each* $y \in (-1, 1)$ corresponds to some $x \in E^1$. (How about ± 1?)

To show that f is increasing, consider separately the three cases $x < 0 < x'$, $x < x' < 0$ and $0 < x < x'$ (also for infinite x and x').]

7. Continuing Problems 5 and 6, consider (E^1, ρ') as a subspace of (E^*, ρ') with ρ' as in Problem 5. Show that globes in (E^1, ρ') are exactly all open intervals in E^*. For example, $(0, 1)$ is a globe. What are its center and radius under ρ' and under the *standard* metric ρ?

8. Metrize the closed interval $[0, +\infty]$ in E^* by setting

$$\rho(x, y) = \left|\frac{1}{1+x} - \frac{1}{1+y}\right|,$$

with the conventions $1 + (+\infty) = +\infty$ and $1/(+\infty) = 0$. Verify the metric axioms. Describe $G_p(1)$ for arbitrary $p \geq 0$.

9. Prove that if ρ is a metric for S, then another metric ρ' for S is given by

(i) $\rho'(x, y) = \min\{1, \rho(x, y)\}$;

(ii) $\rho'(x, y) = \dfrac{\rho(x, y)}{1 + \rho(x, y)}$.

In case (i), show that globes $G_p(\varepsilon)$ of radius $\varepsilon \leq 1$ are the same under ρ and ρ'. In case (ii), prove that any $G_p(\varepsilon)$ in (S, ρ) is also a globe $G_p(\varepsilon')$ in (S, ρ') of radius

$$\varepsilon' = \frac{\varepsilon}{1+\varepsilon},$$

and any globe of radius $\varepsilon' < 1$ in (S, ρ') is also a globe in (S, ρ). (Find the converse formula for ε as well!)

[Hint for the triangle inequality in (ii): Let $a = \rho(x, z)$, $b = \rho(x, y)$, and $c = \rho(y, z)$, so that $a \leq b + c$. The *required* inequality is

$$\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

Simplify it and show that it *follows* from $a \leq b + c$.]

10. Prove that if (X, ρ') and (Y, ρ'') are metric spaces, then a metric ρ for the set $X \times Y$ is obtained by setting, for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$,

(i) $\rho((x_1, y_1), (x_2, y_2)) = \max\{\rho'(x_1, x_2), \rho''(y_1, y_2)\}$; or

(ii) $\rho((x_1, y_1), (x_2, y_2)) = \sqrt{\rho'(x_1, x_2)^2 + \rho''(y_1, y_2)^2}$.

[Hint: For brevity, put $\rho'_{12} = \rho'(x_1, x_2)$, $\rho''_{12} = \rho''(y_1, y_2)$, etc. The triangle inequality in (ii),

$$\sqrt{(\rho'_{13})^2 + (\rho''_{13})^2} \leq \sqrt{(\rho'_{12})^2 + (\rho''_{12})^2} + \sqrt{(\rho'_{23})^2 + (\rho''_{23})^2},$$

is verified by squaring both sides, *isolating the remaining square root* on the right side, simplifying, and squaring again. Simplify by using the triangle inequalities valid in X and Y, i.e.,

$$\rho'_{13} \leq \rho'_{12} + \rho'_{23} \text{ and } \rho''_{13} \leq \rho''_{12} + \rho''_{23}.$$

Reverse all steps, so that the required inequality becomes the *last* step.]

11. Prove that

$$|\rho(y, z) - \rho(x, z)| \leq \rho(x, y)$$

in *any* metric space (S, ρ).

[*Caution*: The formula $\rho(x, y) = |x - y|$, valid in E^n, cannot be used in (S, ρ). Why?]

12. Prove that

$$\rho(p_1, p_2) + \rho(p_2, p_3) + \cdots + \rho(p_{n-1}, p_n) \geq \rho(p_1, p_n).$$

[Hint: Use induction.]

§12. Open and Closed Sets. Neighborhoods

I. Let A be an *open* globe in (S, ρ) or an *open* interval $(\bar{a}, \bar{b})$ in E^n. Then *every* $p \in A$ can be enclosed in a small globe $G_p(\delta) \subseteq A$ (Figures 7 and 8). (This would fail for "boundary" points; but there are none inside an *open* G_q or $(\bar{a}, \bar{b})$.)

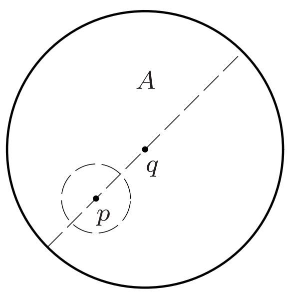

FIGURE 7

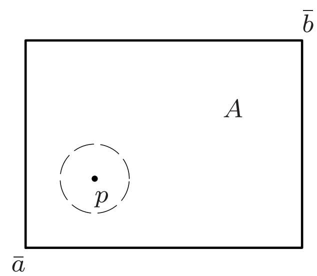

FIGURE 8

This suggests the following ideas, for *any* (S, ρ).

Definition 1.

A point p is said to be *interior* to a set $A \subseteq (S, \rho)$ iff A contains some G_p; i.e., p, *together with some globe* G_p, belongs to A. We then also say that A is a *neighborhood* of p. The set of all interior points of A ("the *interior of* A") is denoted A^0. Note: $\emptyset^0 = \emptyset$ and $S^0 = S$.[1]

Definition 2.

A set $A \subseteq (S, \rho)$ is said to be *open* iff A coincides with its interior $(A^0 = A)$. *Such are* $\emptyset$ *and* S.

[1] Indeed, $\emptyset$ has no points at all, and hence no interior points; i.e., $\emptyset^0$ is void. On the other hand, S contains *any* G_p. Thus *any* p is interior to S; i.e., $S^0 = S$.

Examples.

(1) As noted above, an *open globe* $G_q(r)$ has *interior* points only, and thus is an *open set* in the sense of Definition 2. (See Problem 1 for a proof.)

(2) The same applies to an *open interval* $(\bar{a}, \bar{b})$ in E^n. (See Problem 2.)

(3) The interior of any interval in E^n never includes its endpoints $\bar{a}$ and $\bar{b}$. In fact, it coincides with the *open* interval $(\bar{a}, \bar{b})$. (See Problem 4.)

(4) The set R of all rationals in E^1 has no interior points at all ($R^0 = \emptyset$) because it cannot contain *any* $G_p = (p - \varepsilon, p + \varepsilon)$. Indeed, any such G_p contains *irrationals* (see Chapter 2, §§11–12, Problem 5), so it is not *entirely* contained in R.

Theorem 1 (Hausdorff property[2]). *Any two points p and q $(p \neq q)$ in (S, ρ) are centers of two disjoint globes.*

More precisely,

$$(\exists\, \varepsilon > 0) \quad G_p(\varepsilon) \cap G_q(\varepsilon) = \emptyset.$$

Proof. As $p \neq q$, we have $\rho(p, q) > 0$ by metric axiom (i′). Thus we may put

$$\varepsilon = \frac{1}{2}\rho(p, q) > 0.$$

It remains to show that with this ε, $G_p(\varepsilon) \cap G_q(\varepsilon) = \emptyset$.

Seeking a contradiction, suppose this fails. Then there is $x \in G_p(\varepsilon) \cap G_q(\varepsilon)$ so that $\rho(p, x) < \varepsilon$ and $\rho(x, q) < \varepsilon$. By the triangle law,

$$\rho(p, q) \leq \rho(p, x) + \rho(x, q) < \varepsilon + \varepsilon = 2\varepsilon; \text{ i.e., } \rho(p, q) < 2\varepsilon,$$

which is impossible since $\rho(p, q) = 2\varepsilon$. □

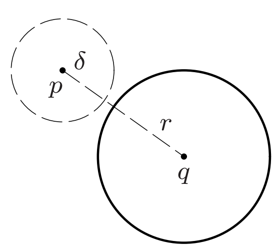

Figure 9

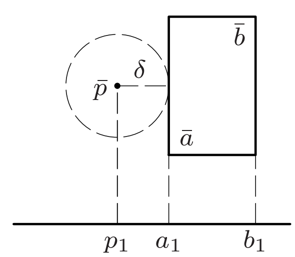

Figure 10

Note. A look at Figure 9 explains the idea of this proof, namely, to obtain two disjoint globes of *equal* radius, it suffices to choose $\varepsilon \leq \frac{1}{2}\rho(p, q)$. *The reader is advised to use such diagrams in E^2* as a guide.

II. We can now define *closed* sets in terms of open sets.

[2] Named after Felix Hausdorff.

Definition 3.

A set $A \subseteq (S, \rho)$ is said to be *closed* iff its complement $-A = S - A$ is *open*, i.e., has interior points only.

That is, each $p \in -A$ (*outside* A) is in some globe $G_p \subseteq -A$ *so that*

$$A \cap G_p = \emptyset.$$

Examples (continued)**.**

(5) The sets $\emptyset$ and S are closed, for their complements, S and $\emptyset$, are open, as noted above. Thus a set may be *both closed and open* ("*clopen*").

(6) All closed globes in (S, ρ) and all closed intervals in E^n are *closed sets* by Definition 3. Indeed (see Figures 9 and 10), if $A = \overline{G}_q(r)$ or $A = [\bar{a}, \bar{b}]$, then any point p *outside* A can be enclosed in a globe $G_p(\delta)$ *disjoint* from A; so, by Definition 3, A is closed (see Problem 12).

(7) A *one-point* set $\{q\}$ (also called "*singleton*") in (S, ρ) is always closed, for any p *outside* $\{q\}$ $(p \neq q)$ is in a globe disjoint from $\{q\}$ by Theorem 1.

In a *discrete space* (§11, Example (3)), $\{q\}$ is also open since it is an *open globe*, $\{q\} = G_q(\frac{1}{2})$ (why?); so it is "clopen." Hence, in such a space, *all sets are "clopen"*. For $p \in A$ implies $\{p\} = G_p(\frac{1}{2}) \subseteq A$; similarly for $-A$. Thus A and $-A$ have *interior* points only, so *both* are open.

(8) The interval $(a, b]$ in E^1 is *neither open nor closed*. (Why?)

***III.** (The rest of this section may be deferred until Chapter 4, §10.)

Theorem 2. *The union of any finite or infinite family of open sets* A_i $(i \in I)$, *denoted*

$$\bigcup_{i \in I} A_i,$$

is open itself. So also is

$$\bigcap_{i=1}^{n} A_i$$

for finitely many open sets. (*This fails for* infinitely many *sets* A_i; *see Problem* 11 *below.*)

Proof. We must show that any point p of $A = \bigcup_i A_i$ is *interior* to A.

Now if $p \in \bigcup_i A_i$, p is in some A_i, and it is an *interior* point of A_i (for A_i is *open*, by assumption). Thus there is a globe

$$G_p \subseteq A_i \subseteq A,$$

as required.

For finite *intersections*, it suffices to consider *two* open sets A and B (for n sets, all then follows by induction). We must show that each $p \in A \cap B$ is *interior* to $A \cap B$.

Now as $p \in A$ and A is *open*, we have some $G_p(\delta') \subseteq A$. Similarly, there is $G_p(\delta'') \subseteq B$. Then the *smaller* of the two globes, call it G_p, is in *both* A and B, so

$$G_p \subseteq A \cap B$$

and p is interior to $A \cap B$, indeed. □

Theorem 3. *If the sets A_i $(i \in I)$ are closed, so is*

$$\bigcap_{i \in I} A_i$$

(*even for infinitely many sets*). *So also is*

$$\bigcup_{i=1}^{n} A_i$$

for finitely many closed sets A_i. (Again, this fails for infinitely many *sets A_i.)*

Proof. Let $A = \bigcap_{i \in I} A_i$. To prove that A is closed, we show that $-A$ is open.

Now by set theory (see Chapter 1, §§1–3, Theorem 2),

$$-A = -\bigcap_i A_i = \bigcup_i (-A_i),$$

where the $(-A_i)$ are *open* (for the A_i are closed). Thus by Theorem 2, $-A$ *is open*, as required.

The second assertion (as to $\bigcup_{i=1}^{n} A_i$) follows quite similarly. □

Corollary 1. *A nonempty set $A \subseteq (S, \rho)$ is open iff A is a union of open globes.*

For if A is such a union, it is open by Theorem 2. Conversely, if A is open, then each $p \in A$ is in some $G_p \subseteq A$. All such G_p $(p \in A)$ cover *all* of A, so $A \subseteq \bigcup_{p \in A} G_p$. Also, $\bigcup_{p \in A} G_p \subseteq A$ since all G_p are in A. Thus

$$A = \bigcup_{p \in A} G_p.$$

Corollary 2. *Every finite set F in a metric space (S, ρ) is closed.*

Proof. If $F = \emptyset$, F is closed by Example (5). If $F \neq \emptyset$, let

$$F = \{p_1, \ldots, p_n\} = \bigcup_{k=1}^{n} \{p_k\}.$$

Now by Example (7), each $\{p_k\}$ is closed; hence so is F by Theorem 3. □

Note. The family of *all* open sets in a given space (S, ρ) is denoted by $\mathcal{G}$; that of all closed sets, by $\mathcal{F}$. Thus "$A \in \mathcal{G}$" means that A is *open*; "$A \in \mathcal{F}$" means that A is *closed*. By Theorems 2 and 3, we have

$$(\forall\, A,\, B \in \mathcal{G}) \quad A \cup B \in \mathcal{G} \text{ and } A \cap B \in \mathcal{G};$$

similarly for $\mathcal{F}$. This is a kind of "closure law." We say that $\mathcal{F}$ and $\mathcal{G}$ are "*closed under finite unions and intersections*."

In conclusion, consider any *subspace* (A, ρ) of (S, ρ). As we know from §11, it is a metric space itself, so *it has its own open and closed sets* (which must consist of points of A only). We shall now show that they are obtained from those of (S, ρ) *by intersecting the latter sets with* A.

Theorem 4. *Let* (A, ρ) *be a subspace of* (S, ρ). *Then the open* (*closed*) *sets in* (A, ρ) *are exactly all sets of the form* $A \cap U$, *with* U *open* (*closed*) *in* S.

Proof. Let G be open *in* (A, ρ). By Corollary 1, G is the union of some open globes G_i^* $(i \in I)$ *in* (A, ρ). (For brevity, we omit the centers and radii; we also omit the trivial case $G = \emptyset$.)

As was shown in §11, however, $G_i^* = A \cap G_i$, where G_i is an open globe *in* (S, ρ). Thus

$$G = \bigcup_i G_i^* = \bigcup_i (A \cap G_i) = A \cap \bigcup_i G_i,$$

by set theory (see Chapter 1, §§1–3, Problem 9).

Again by Corollary 1, $U = \bigcup_i G_i$ is an open set *in* (S, ρ). Thus G has the form

$$A \cap \bigcup_i G_i = A \cap U,$$

with U open in S, as asserted.

Conversely, assume the latter, and let $p \in G$. Then $p \in A$ and $p \in U$. As U is open in (S, ρ), there is a globe G_p in (S, ρ) such that $p \in G_p \subseteq U$. As $p \in A$, we have

$$p \in A \cap G_p \subseteq A \cap U.$$

However, $A \cap G_p$ *is a globe in* (A, ρ), call it G_p^*. Thus

$$p \in G_p^* \subseteq A \cap U = G;$$

i.e., p is an *interior* point of G *in* (A, ρ). We see that each $p \in G$ is interior to G, as a set in (A, ρ), so G is open in (A, ρ).

This proves the theorem for *open* sets. Now let F be closed in (A, ρ). Then by Definition 3, $A - F$ is open in (A, ρ). (Of course, when working in (A, ρ), we *replace* S *by* A in taking complements.) Let $G = A - F$, so $F = A - G$, and G is open in (A, ρ). By what was shown above, $G = A \cap U$ with U open in S.

Thus

$$F = A - G = A - (A \cap U) = A - U = A \cap (-U)$$

by set theory. Here $-U = S - U$ is *closed in* (S, ρ) since U is open there. Thus $F = A \cap (-U)$, as required.

The proof of the converse (for closed sets) is left as an exercise. □

Problems on Neighborhoods, Open and Closed Sets

⇒1. Verify Example (1).

[Hint: Given $p \in G_q(r)$, let

$$\delta = r - \rho(p, q) > 0. \quad \text{(Why } > 0\text{?)}$$

Use the triangle law to show that

$$x \in G_p(\delta) \Rightarrow \rho(x, q) < r \Rightarrow x \in G_q(r).]$$

⇒2. Check Example (2); see Figure 8.

[Hint: If $\bar{p} \in (\bar{a}, \bar{b})$, choose δ less than the $2n$ numbers

$$p_k - a_k \text{ and } b_k - p_k, \quad k = 1, \ldots, n;$$

then show that $G_{\bar{p}}(\delta) \subseteq (\bar{a}, \bar{b})$.]

3. Prove that if $\bar{p} \in G_{\bar{q}}(r)$ in E^n, then $G_{\bar{q}}(r)$ contains a cube $[\bar{c}, \bar{d}]$ with $\bar{c} \neq \bar{d}$ and with center $\bar{p}$.

[Hint: By Example (1), there is $G_{\bar{p}}(\delta) \subseteq G_{\bar{q}}(r)$. Inscribe in $G_{\bar{p}}(\frac{1}{2}\delta)$ a cube of diagonal δ. *Find its edge-length* $(\delta/\sqrt{n})$. Then use it to find the coordinates of the endpoints, $\bar{c}$ and $\bar{d}$ (given $\bar{p}$, the center). Prove that $[\bar{c}, \bar{d}] \subseteq G_{\bar{p}}(\delta)$.]

4. Verify Example (3).

[Hint: To show that no interior points of $[\bar{a}, \bar{b}]$ are *outside* $(\bar{a}, \bar{b})$, let $\bar{p} \notin (\bar{a}, \bar{b})$. Then at least one of the inequalities $a_k < p_k$ or $p_k < b_k$ *fails*. (Why?) Let it be $a_1 < p_1$, say, so $p_1 \leq a_1$.

Now take *any* globe $G_{\bar{p}}(\delta)$ about $\bar{p}$ and prove that it is *not* contained in $[\bar{a}, \bar{b}]$ (so $\bar{p}$ *cannot* be an interior point). For this purpose, as in Problem 3, show that $G_{\bar{p}}(\delta) \supseteq [\bar{c}, \bar{d}]$ with $c_1 < p_1 \leq a_1$. Deduce that $\bar{c} \in G_{\bar{p}}(\delta)$, but $\bar{c} \notin [\bar{a}, \bar{b}]$; so $G_{\bar{p}}(\delta) \not\subseteq [\bar{a}, \bar{b}]$.]

5. Prove that each open globe $G_{\bar{q}}(r)$ in E^n is a union of *cubes* (which can be made open, closed, half-open, etc., as desired). Also, show that each open interval $(\bar{a}, \bar{b}) \neq \emptyset$ in E^n is a union of open (or closed) globes.

[Hint for the first part: By Problem 3, each $\bar{p} \in G_{\bar{q}}(r)$ is in a cube $C_p \subseteq G_{\bar{q}}(r)$. Show that $G_{\bar{q}}(r) = \bigcup C_p$.]

6. Show that every globe in E^n contains *rational* points, i.e., those with rational coordinates only (we express it by saying that the set R^n of such points is *dense* in E^n); similarly for the set I^n of *irrational* points (those with irrational coordinates).

[Hint: First check it with globes replaced by *cubes* $(\bar{c}, \bar{d})$; see §7, Corollary 3. Then use Problem 3 above.]

7. Prove that if $\bar{x} \in G_{\bar{q}}(r)$ in E^n, there is a *rational* point $\bar{p}$ (Problem 6) and a *rational* number $\delta > 0$ such that $\bar{x} \in G_{\bar{p}}(\delta) \subseteq G_{\bar{q}}(r)$. Deduce that each globe $G_{\bar{q}}(r)$ in E^n is a union of *rational* globes (those with rational centers and radii). Similarly, show that $G_{\bar{q}}(r)$ is a union of *intervals* with rational endpoints.

[Hint for the first part: Use Problem 6 and Example (1).]

8. Prove that if the points $p_1, \ldots, p_n$ in (S, ρ) are *distinct*, there is an $\varepsilon > 0$ such that the globes $G(p_k; \varepsilon)$ are disjoint from each other, for $k = 1, 2, \ldots, n$.

9. Do Problem 7, with $G_{\bar{q}}(r)$ replaced by an arbitrary *open set* $G \neq \emptyset$ in E^n.

10. Show that every open set $G \neq \emptyset$ in E^n is infinite (*even uncountable; see Chapter 1, §9).

[Hint: Choose $G_{\bar{q}}(r) \subseteq G$. By Problem 3, $G_{\bar{p}}(r) \supset L[\bar{c}, \bar{d}]$, a line segment.]

11. Give examples to show that an infinite intersection of open sets may not be open, and an infinite union of closed sets may not be closed.

[Hint: Show that

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

and

$$\bigcup_{n=2}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right] = (0, 1).]$$

12. Verify Example (6) as suggested in Figures 9 and 10.

[Hints: (i) For $\overline{G}_q(r)$, take

$$\delta = \rho(p, q) - r > 0. \quad (\text{Why} > 0?)$$

(ii) If $\bar{p} \notin [\bar{a}, \bar{b}]$, at least one of the $2n$ inequalities $a_k \leq p_k$ or $p_k \leq b_k$ fails (why?), say, $p_1 < a_1$. Take $\delta = a_1 - p_1$.

In both (i) and (ii) *prove* that $A \cap G_p(\delta) = \emptyset$ (proceed as in Theorem 1).]

***13.** Prove the last parts of Theorems 3 and 4.

***14.** Prove that A^0, the interior of A, is the union of all open globes contained in A (assume $A^0 \neq \emptyset$). Deduce that A^0 is an open set, the *largest* contained in A.[3]

***15.** For sets $A, B \subseteq (S, \rho)$, prove that

(i) $(A \cap B)^0 = A^0 \cap B^0$;

(ii) $(A^0)^0 = A^0$; and

(iii) if $A \subseteq B$ then $A^0 \subseteq B^0$.

[3] That is, the one that contains all other open subsets of A.

[Hint for (ii): A^0 is *open* by Problem 14.]

16. Is $A^0 \cup B^0 = (A \cup B)^0$?
[Hint: See Example (4). Take $A = R$, $B = E^1 - R$.]

17. Prove that if M and N are neighborhoods of p in (S, ρ), then

(a) $p \in M \cap N$;

(b) $M \cap N$ is a neighborhood of p;

*(c) so is M^0; and

(d) so also is each set $P \subseteq S$ such that $P \supseteq M$ or $P \supseteq N$.

[Hint for (c): See Problem 14.]

18. The *boundary* of a set $A \subseteq (S, \rho)$ is defined by

$$\operatorname{bd} A = -[A^0 \cup (-A)^0];$$

thus it consists of points that *fail* to be interior in A or in $-A$.

Prove that the following statements are true:

(i) $S = A^0 \cup \operatorname{bd} A \cup (-A)^0$, all disjoint.

(ii) $\operatorname{bd} S = \emptyset$, $\operatorname{bd} \emptyset = \emptyset$.

*(iii) A is open iff $A \cap \operatorname{bd} A = \emptyset$; A is closed iff $A \supseteq \operatorname{bd} A$.

(iv) In E^n,

$$\operatorname{bd} G_{\bar{p}}(r) = \operatorname{bd} \overline{G}_{\bar{p}}(r) = S_{\bar{p}}(r)$$

(the sphere with center $\bar{p}$ and radius r). Is this true in *all* metric spaces?
[Hint: Consider $G_p(\frac{1}{2})$ in a *discrete* space; see §11, Example (3).]

(v) In E^n, if $(a, b) \neq \emptyset$, then

$$\operatorname{bd}(\bar{a}, \bar{b}] = \operatorname{bd}[\bar{a}, \bar{b}) = \operatorname{bd}(\bar{a}, \bar{b}) = \operatorname{bd}[\bar{a}, \bar{b}] = [\bar{a}, \bar{b}] - (\bar{a}, \bar{b}).$$

(vi) In E^n, $(R^n)^0 = \emptyset$; hence $\operatorname{bd} R^n = E^n$ (R^n as in Problem 6).

19. Verify Example (8) for intervals *in* E^n.

§13. Bounded Sets. Diameters

I. Geometrically, the diameter of a closed globe in E^n could be defined as the maximum distance between two of its points. In an *open* globe in E^n, there is no "maximum" distance (why?), but we still may consider the *supremum* of all distances inside the globe. Moreover, this makes sense in *any* set $A \subseteq (S, \rho)$. Thus we accept it as a general definition, for *any* such set.

Definition 1.

The *diameter* of a set $A \neq \emptyset$ in a metric space (S, ρ), denoted dA, is the supremum (in E^*) of all distances $\rho(x, y)$, with $x, y \in A$;[1] in symbols,

$$dA = \sup_{x, y \in A} \rho(x, y).$$

If $A = \emptyset$, we put $dA = 0$. If $dA < +\infty$, A is said to be *bounded* (in (S, ρ)).

Equivalently, we could define a bounded set as in the statement of the following theorem.

Theorem 1. *A set $A \subseteq (S, \rho)$ is bounded iff A is contained in some globe. If so, the center p of this globe can be chosen at will.*

Proof. If $A = \emptyset$, all is trivial.

Thus let $A \neq \emptyset$; let $q \in A$, and choose *any* $p \in S$. Now if A is bounded, then $dA < +\infty$, so we can choose a real $\varepsilon > \rho(p, q) + dA$ as a suitable radius for a globe $G_p(\varepsilon) \supseteq A$ (see Figure 11 for motivation). Now if $x \in A$, then by the definition of dA, $\rho(q, x) \leq dA$; so by the triangle law,

$$\begin{aligned} \rho(p, x) &\leq \rho(p, q) + \rho(q, x) \\ &\leq \rho(p, q) + dA < \varepsilon; \end{aligned}$$

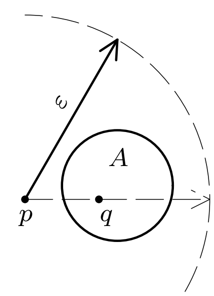

FIGURE 11

i.e., $x \in G_p(\varepsilon)$. Thus $(\forall x \in A)\ x \in G_p(\varepsilon)$, as required.

Conversely, if $A \subseteq G_p(\varepsilon)$, then any $x, y \in A$ are also in $G_p(\varepsilon)$; so $\rho(x, p) < \varepsilon$ and $\rho(p, y) < \varepsilon$, whence

$$\rho(x, y) \leq \rho(x, p) + \rho(p, y) < \varepsilon + \varepsilon = 2\varepsilon.$$

Thus 2ε is an upper bound of all $\rho(x, y)$ with $x, y \in A$. Therefore,

$$dA = \sup \rho(x, y) \leq 2\varepsilon < +\infty;$$

i.e., A is bounded, and all is proved. □

As a special case we obtain the following.

Theorem 2. *A set $A \subseteq E^n$ is bounded iff there is a real $K > 0$ such that*

$$(\forall \bar{x} \in A) \quad |\bar{x}| < K$$

*(similarly in C^n *and other normed spaces).*

[1] Recall that the supremum *always* exists in E^* (finite or not); see Chapter 2, §13.

Proof. By Theorem 1 (choosing $\bar{0}$ for p), A is bounded iff A is contained in some globe $G_{\bar{0}}(\varepsilon)$ about $\bar{0}$. That is,

$$(\forall\, \bar{x} \in A) \quad \bar{x} \in G_{\bar{0}}(\varepsilon) \text{ or } \rho(\bar{x}, \bar{0}) = |\bar{x}| < \varepsilon.$$

Thus ε is the required K. (*The proof for normed spaces is the same.) □

Note 1. In E^1, this means that

$$(\forall\, x \in A) \quad -K < x < K;$$

i.e., A is bounded by $-K$ and K. This agrees with our former definition, given in Chapter 2, §§8–9.

Caution: *Upper* and *lower* bounds are not defined in (S, ρ), in general.

Examples.

(1) $\emptyset$ is bounded, with $d\emptyset = 0$, by definition.

(2) Let $A = [\bar{a}, \bar{b}]$ in E^n, with $d = \rho(\bar{a}, \bar{b})$ its diagonal. By Corollary 1 in §7, d is the largest distance in A. In nonclosed intervals, we still have

$$d = \sup_{x,\, y \in A} \rho(x, y) = dA < +\infty \text{ (see Problem 10(ii)).}$$

Thus *all intervals in* E^n *are bounded*.

(3) Each globe $G_p(\varepsilon)$ in (S, ρ) is bounded, with $dG_p(\varepsilon) \le 2\varepsilon < +\infty$, as was shown in the proof of Theorem 1. See, however, Problems 5 and 6 below.

(4) *All of* E^n is not bounded, under the standard metric, for if E^n had a finite diameter d, no distance in E^n would exceed d; but $\rho(-d\bar{e}_1, d\bar{e}_1) = 2d$, a contradiction!

(5) On the other hand, under the *discrete metric* (§11, Example (3)), *any* set (even the entire space) is contained in $G_p(3)$ and hence bounded. The same applies to the metric ρ' defined for E^* in Problem 5 of §11, since distances under that metric never exceed 2, and so $E^* \subseteq G_p(3)$ for any choice of p.

Note 2. This shows that *boundedness depends on the metric* ρ. A set may be bounded under one metric and not bounded under another. A metric ρ is said to be *bounded* iff *all* sets are bounded under ρ (as in Example (5)).

Problem 9 of §11 shows that *any metric* ρ *can be transformed into a bounded one*, even preserving all sufficiently small globes; in part (i) of the problem, even the radii remain the same if they are ≤ 1.

Note 3. An idea similar to that of diameter is often used to define distances between *sets*. If $A \ne \emptyset$ and $B \ne \emptyset$ in (S, ρ), we define $\rho(A, B)$ to be the *infimum* of all distances $\rho(x, y)$, with $x \in A$ and $y \in B$. In particular, if $B = \{p\}$ (a singleton), we write $\rho(A, p)$ for $\rho(A, B)$. Thus

$$\rho(A, p) = \inf_{x \in A} \rho(x, p).$$

II. The definition of boundedness extends, in a natural manner, to sequences and functions. We briefly write $\{x_m\} \subseteq (S, \rho)$ for a sequence of points in (S, ρ), and $f\colon A \to (S, \rho)$ for a mapping of an *arbitrary* set A into the space S. Instead of "infinite sequence with general term x_m," we say "the sequence x_m."

Definition 2.

A sequence $\{x_m\} \subseteq (S, \rho)$ is said to be *bounded* iff its range is bounded in (S, ρ), i.e., iff all its terms x_m are contained in some globe in (S, ρ).

In E^n, this means (by Theorem 2) that

$$(\forall\, m) \quad |x_m| < K$$

for some fixed $K \in E^1$.[2]

Definition 3.

A function $f\colon A \to (S, \rho)$ is said to be *bounded on a set* $B \subseteq A$ iff the image set $f[B]$ is bounded in (S, ρ); i.e. iff all function values $f(x)$, with $x \in B$, are in some globe in (S, ρ).

In E^n, this means that

$$(\forall\, x \in B) \quad |f(x)| < K$$

for some fixed $K \in E^1$.[2]

If $B = A$, we simply say that f is *bounded*.

Note 4. If $S = E^1$ or $S = E^*$, we may also speak of *upper* and *lower* bounds. It is customary to call $\sup f[B]$ also the *supremum of f on B* and denote it by symbols like

$$\sup_{x \in B} f(x) \text{ or } \sup\{f(x) \mid x \in B\}.$$

In the case of sequences, we often write $\sup_m x_m$ or $\sup x_m$ instead; similarly for infima, maxima, and minima.

Examples.

(a) The sequence

$$x_m = \frac{1}{m} \quad \text{in } E^1$$

is bounded since all terms x_m are in the interval $(0, 2) = G_1(1)$. We have $\inf x_m = 0$ and $\sup x_m = \max x_m = 1$.

[2] *Similarly in C^n and other normed spaces.

(b) The sequence

$$x_m = m \quad \text{in } E^1$$

is bounded *below* (by 1) but not above. We have $\inf x_m = \min x_m = 1$ and $\sup x_m = +\infty$ (in E^*).

(c) Define $f\colon E^1 \to E^1$ by

$$f(x) = 2x.$$

This map is bounded on each finite interval $B = (a, b)$ since $f[B] = (2a, 2b)$ is itself an interval and hence bounded. However, f is not bounded on *all* of E^1 since $f[E^1] = E^1$ is not a bounded set.

(d) Under a bounded metric ρ, all functions $f\colon A \to (S, \rho)$ are bounded.

(e) The so-called *identity map on* S, $f\colon S \to (S, \rho)$, is defined by

$$f(x) = x.$$

Clearly, f carries each set $B \subseteq S$ onto itself; i.e., $f[B] = B$. Thus f is bounded on B iff B is itself a bounded set in (S, ρ).

(f) Define $f\colon E^1 \to E^1$ by

$$f(x) = \sin x.$$

Then $f[E^1] = [-1, 1]$ is a bounded set in the range space E^1. Thus f is bounded on E^1 (briefly, *bounded*).

Problems on Boundedness and Diameters

1. Show that if a set A in a metric space is bounded, so is each subset $B \subseteq A$.

2. Prove that if the sets $A_1, A_2, \ldots, A_n$ in (S, ρ) are bounded, so is

$$\bigcup_{k=1}^{n} A_k.$$

Disprove this for *infinite* unions by a counterexample.

[Hint: By Theorem 1, each A_k is in some $G_p(\varepsilon_k)$, *with one and the same center* p. If the number of the globes is finite, we can put $\max(\varepsilon_1, \ldots, \varepsilon_n) = \varepsilon$, so $G_p(\varepsilon)$ contains *all* A_k. Verify this in detail.]

⇒3. From Problems 1 and 2 show that a set A in (S, ρ) is bounded iff it is contained in a finite union of globes,

$$\bigcup_{k=1}^{n} G(p_k; \varepsilon_k).$$

4. A set A in (S, ρ) is said to be *totally bounded* iff for every $\varepsilon > 0$ (*no matter how small*), A is contained in a finite union of globes of radius ε. By Problem 3, any such set is bounded. Disprove the converse by a counterexample.
[Hint: Take an infinite set in a *discrete* space.]

5. Show that distances between points of a globe $\overline{G}_p(\varepsilon)$ never exceed 2ε. (Use the triangle inequality!) Hence infer that $dG_p(\varepsilon) \le 2\varepsilon$. Give an example where $dG_p(\varepsilon) < 2\varepsilon$. Thus the diameter of a globe may be *less* than twice its radius.
[Hint: Take a globe $G_p(\frac{1}{2})$ in a *discrete* space.]

6. Show that in E^n (*as well as in C^n and any other normed linear space $\neq \{0\}$), the diameter of a globe $G_p(\varepsilon)$ always *equals* 2ε (twice its radius).
[Hint: By Problem 5, 2ε is an upper bound of all $\rho(\bar{x}, \bar{y})$ with $\bar{x}, \bar{y} \in G_p(\varepsilon)$.
To show that there is no *smaller* upper bound, prove that any number

$$2\varepsilon - 2r \quad (r > 0)$$

is *exceeded* by some $\rho(\bar{x}, \bar{y})$; e.g., take $\bar{x}$ and $\bar{y}$ on some line through $\bar{p}$,

$$\bar{x} = \bar{p} + t\vec{u},$$

choosing suitable values for t to get $\rho(\bar{x}, \bar{y}) = |\bar{x} - \bar{y}| > 2\varepsilon - 2r$.]

7. Prove that in E^n, a set A is bounded iff it is contained in an *interval*.

8. Prove that

$$\rho(A, B) \le \rho(A, p) + \rho(p, B).$$

Disprove

$$\rho(A, B) < \rho(A, p) + \rho(p, B)$$

by an example.

9. Find $\sup x_n$, $\inf x_n$, $\max x_n$, and $\min x_n$ (if any) for sequences with general term

(a) n;

(b) $(-1)^n(2 - 2^{2-n})$;

(c) $1 - \dfrac{2}{n}$;

(d) $\dfrac{n(n-1)}{(n+2)^2}$.

Which are bounded in E^1?

10. Prove the following about lines and line segments.

(i) Show that any line segment in E^n is a bounded set, but the entire line is not.

(ii) Prove that the diameter of $L(\bar{a}, \bar{b})$ and of $(\bar{a}, \bar{b})$ equals $\rho(\bar{a}, \bar{b})$.

11. Let $f\colon E^1 \to E^1$ be given by

$$f(x) = \frac{1}{x} \text{ if } x \neq 0, \text{ and } f(0) = 0.$$

Show that f is bounded on an interval $[a, b]$ iff $0 \notin [a, b]$. Is f bounded on $(0, 1)$?

12. Prove the following:

(a) If $A \subseteq B \subseteq (S, \rho)$, then $dA \leq dB$.

(b) $dA = 0$ iff A contains at most one point.

(c) If $A \cap B \neq \emptyset$, then

$$d(A \cup B) \leq dA + dB.$$

Show by an example that this may fail if $A \cap B = \emptyset$.

§14. Cluster Points. Convergent Sequences

Consider the set

$$A = \left\{1, \frac{1}{2}, \ldots, \frac{1}{m}, \ldots\right\};$$

we may as well let A denote the *sequence* $x_m = 1/m$ in E^1.[1] Plotting it on the axis, we observe a remarkable fact: The points x_m "cluster" close to 0, approaching 0 as m increases—see Figure 12.

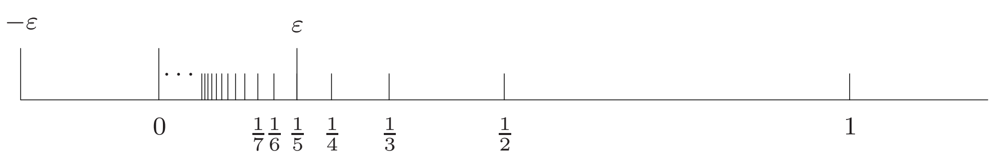

FIGURE 12

To make this more precise, take any globe about 0 in E^1, $G_0(\varepsilon) = (-\varepsilon, \varepsilon)$. No matter how small, it contains *infinitely many* (even *all but finitely many*) points x_m, namely, all from some x_k onward, so that

$$(\forall\, m > k) \quad x_m \in G_0(\varepsilon).$$

Indeed, take $k > 1/\varepsilon$, so $1/k < \varepsilon$. Then

$$(\forall\, m > k) \quad \frac{1}{m} < \frac{1}{k} < \varepsilon;$$

i.e., $x_m \in (-\varepsilon, \varepsilon) = G_0(\varepsilon)$.

This suggests the following generalizations.

[1] "Sequence" means "*infinite* sequence"; m, n, k denote integers > 0.

Definition 1.

A set, or sequence, $A \subseteq (S, \rho)$ is said to *cluster* at a point $p \in S$ (not necessarily $p \in A$), and p is called its *cluster point* or *accumulation point*, iff every globe G_p about p contains infinitely many points (respectively, terms) of A. (Thus *only infinite sets can cluster*.)

Note 1. In sequences (unlike *sets*) an infinitely repeating term counts as *infinitely many* terms. For example, the sequence $0, 1, 0, 1, \ldots$ clusters at 0 and 1 (why?); but its *range*, $\{0, 1\}$, has *no* cluster points (being *finite*). This distinction is, however, irrelevant if all terms x_m are *distinct*, i.e., different from each other. Then we may treat sequences and sets alike.

Definition 2.

A sequence $\{x_m\} \subseteq (S, \rho)$ is said to *converge* or *tend* to a point p in S, and p is called its *limit*, iff every globe $G_p(\varepsilon)$ about p (no matter how small) contains *all but finitely many terms* x_m.[2] In symbols,

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, m > k) \quad x_m \in G_p(\varepsilon), \text{ i.e., } \rho(x_m, p) < \varepsilon. \tag{1}$$

If such a p exists, we call $\{x_m\}$ a *convergent* sequence (in (S, ρ)); otherwise, a *divergent* one. The notation is

$$x_m \to p, \text{ or } \lim x_m = p, \text{ or } \lim_{m \to \infty} x_m = p.$$

In E^n,[3] $\rho(\bar{x}_m, \bar{p}) = |\bar{x}_m - \bar{p}|$; thus formula (1) turns into

$$\bar{x}_m \to \bar{p} \text{ in } E^n \text{ iff } (\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, m > k) \quad |\bar{x}_m - \bar{p}| < \varepsilon. \tag{2}$$

Since "all but finitely many" (as in Definition 2) *implies* "infinitely many" (as in Definition 1), *any limit is also a cluster point*. Moreover, we obtain the following result.

Corollary 1. *If $x_m \to p$, then p is the unique cluster point of $\{x_m\}$. (Thus a sequence with two or more cluster points, or none at all, diverges.)*

For if $p \neq q$, the Hausdorff property (Theorem 1 of §12) yields an ε such that

$$G_p(\varepsilon) \cap G_q(\varepsilon) = \emptyset.$$

As $x_m \to p$, $G_p(\varepsilon)$ leaves out at most finitely many x_m, and *only these* can possibly be in $G_q(\varepsilon)$. (Why?) Thus q *fails* to satisfy Definition 1 and hence is no cluster point. Hence $\lim x_m$ (if it exists) is unique.

---

[2] That is, $G_p(\varepsilon)$ *leaves out* at most finitely many terms x_m, say, $x_1, x_2, \ldots, x_k$, whereas in Definition 1, $G_p(\varepsilon)$ may leave out even infinitely many points of A.

[3] *Similarly for sequences in C^n and in other normed spaces (§10).

Corollary 2.

(i) *We have* $x_m \to p$ *in* (S, ρ) *iff* $\rho(x_m, p) \to 0$ *in* E^1.

Hence

(ii) $\bar{x}_m \to \bar{p}$ *in* E^n *iff* $|\bar{x}_m - \bar{p}| \to 0$ *and*

(iii) $\bar{x}_m \to \bar{0}$ *in* E^n *iff* $|\bar{x}_m| \to 0$.

Proof. By (2), we have $\rho(x_m, p) \to 0$ in E^1 if

$$(\forall \varepsilon > 0)\ (\exists k)\ (\forall m > k) \quad |\rho(x_m, p) - 0| = \rho(x_m, p) < \varepsilon.$$

By (1), however, this means that $x_m \to p$, proving our first assertion. The rest easily follows from it, since $\rho(\bar{x}_m, \bar{p}) = |\bar{x}_m - \bar{p}|$ in E^n. $\square$

Corollary 3. *If* x_m *tends to* p, *then so does each subsequence* x_{m_k}.

For $x_m \to p$ means that each G_p leaves out at most finitely many x_m. This certainly still holds if we *drop* some terms, passing to $\{x_{m_k}\}$.

Note 2. A similar argument shows that *the convergence or divergence of* $\{x_m\}$*, and its limit or cluster points, are not affected by dropping or adding a finite number of terms*; similarly for cluster points of *sets*. For example, if $\{x_m\}$ tends to p, so does $\{x_{m+1}\}$ (the same sequence without x_1).

We leave the following two corollaries as exercises.

Corollary 4. *If* $\{x_m\}$ *splits into two subsequences, each tending to the same limit* p, *then also* $x_m \to p$.

Corollary 5. *If* $\{x_m\}$ *converges in* (S, ρ), *it is bounded there.* (*See Problem* 4.)

Of course, the convergence or divergence of $\{x_m\}$ and its clustering depend on the metric ρ and the space S. Our theory applies to *any* (S, ρ). In particular, *it applies to* E^**, with the metric* ρ' *of Problem* 5 *in* §11. Recall that under that metric, globes about $\pm\infty$ have the form $(a, +\infty]$ and $[-\infty, a)$, respectively. Thus limits and cluster points in (E^*, ρ') coincide with those defined in Chapter 2, §13, (formulas (1)–(3) and Definition 2 there).[4] Our theory then applies to infinite limits as well, and generalizes Chapter 2, §13.

Examples.

(a) Let

$$x_m = p \quad \text{for all } m$$

(such sequences are called *constant*). As $p \in G_p$, any G_p contains *all* x_m. Thus $x_m \to p$, by Definition 2. We see that *each constant sequence converges to the common value of its terms*.

[4] The second part of Chapter 2, §13, should be reviewed at this stage.

(b) In our introductory example, we showed that

$$\lim_{m\to\infty}\frac{1}{m}=0 \quad \text{in } E^1$$

and that 0 is the (unique) cluster point of the *set* $A=\{1, \frac{1}{2}, \dots\}$. Here $0 \notin A$.

(c) The sequence

$$0, 1, 0, 1, \dots$$

has *two* cluster points, 0 and 1, so it *diverges* by Corollary 1. (It "oscillates" from 0 to 1.) This shows that *a bounded sequence may diverge*. The converse to Corollary 5 fails.

(d) The sequence

$$x_m = m$$

(or the set N of all naturals) has *no* cluster points in E^1, for a globe of radius $< \frac{1}{2}$ (with *any* center $p \in E^1$) contains at most one x_m, and hence *no* p satisfies Definition 1 or 2.

However, $\{x_m\}$ does cluster in (E^*, ρ'), and even has a limit there, namely $+\infty$. (Prove it!)

(e) The set R of all rationals in E^1 clusters at *each* $p \in E^1$. Indeed, *any* globe

$$G_p(\varepsilon) = (p-\varepsilon, p+\varepsilon)$$

contains infinitely many rationals (see Chapter 2, §10, Theorem 3), and this means that each $p \in E^1$ is a cluster point of R.

(f) The sequence

$$1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, \dots \quad (\text{with } x_{2k}=\frac{1}{k} \text{ and } x_{2k-1}=k)$$

has only one cluster point, 0, in E^1; yet it diverges, being unbounded (see Corollary 5). In (E^*, ρ'), it has *two* cluster points, 0 and $+\infty$. (Verify!)

(g) The $\overline{\lim}$ and $\underline{\lim}$ of any sequence in E^* are cluster points (cf. Chapter 2, §13, Theorem 2 and Problem 4). Thus *in E^*, all sequences cluster*.

(h) Let

$$A = [a, b], \quad a < b.$$

Then A clusters exactly at *all its points*, for if $p \in A$, then any globe

$$G_p(\varepsilon) = (p-\varepsilon, p+\varepsilon)$$

overlaps with A (even with (a, b)) and so contains infinitely many points of A, as required. Even the *endpoints a and b* are cluster points of A (and of (a, b), $(a, b]$, and $[a, b)$). On the other hand, no point *outside* A is a cluster point. (Why?)

(i) In a *discrete* space (§11, Example (3)), *no* set can cluster, since small globes, such as $G_p(\frac{1}{2})$, are *singletons*. (Explain!)

Example (h) shows that a set A may *equal* the set of its cluster points (call it A'); i.e.,

$$A = A'.$$

Such sets are said to be *perfect*. Sometimes we have $A \subseteq A'$, $A' \subseteq A$, $A' = S$ (as in Example (e)), or $A' = \emptyset$. We conclude with the following result.

Corollary 6. *A set $A \subseteq (S, \rho)$ clusters at p iff each globe G_p (about p) contains at least one point of A other than p.*[5]

Indeed, assume the latter. Then, in particular, each globe

$$G_p\left(\frac{1}{n}\right), \quad n = 1, 2, \ldots,$$

contains some point of A *other than* p; call it x_n. We can make the x_n *distinct* by choosing each time x_{n+1} *closer* to p than x_n is. It easily follows that each $G_p(\varepsilon)$ contains *infinitely many* points of A (the details are left to the reader), as required. The converse is obvious.

Problems on Cluster Points and Convergence

1. Is the *Archimedean property* (see Chapter 2, §10) involved in the proof that

$$\lim_{m \to \infty} \frac{1}{m} = 0?$$

2. Prove Note 2 and Corollaries 4 and 6.

3. Verify Example (c) in detail.[6]

4. Prove Corollary 5.
[Hint: Fix some $G_p(\varepsilon)$. Use Definition 2. If $G_p(\varepsilon)$ leaves out $x_1, x_2, \ldots, x_k$, take a *larger* radius r greater than

$$\rho(x_m, p), \quad m = 1, 2, \ldots, k.$$

Then the enlarged globe $G_p(r)$ contains *all* x_m. Use Theorem 1 in §13.]

5. Show that $x_m = m$ tends to $+\infty$ in E^*. Does it contradict Corollary 5?

6. Show that E^1 is a perfect set *in* E^1: $E^1 = (E^1)'$. Is E^1 a perfect set *in* E^*? Why?

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[5] This corollary does not apply to cluster points of *sequences*.
[6] In particular, show that there are no *other* cluster points.

⇒**7.** Review Problems 2 and 4 of Chapter 2, §13. (*Do* them if not done before.)

8. Verify Examples (f) and (h).

9. Explain Example (i) in detail.

10. In the following cases find the set A' of all cluster points of A in E^1. Is $A' \subseteq A$? Is $A \subseteq A'$? Is A perfect? Give a precise proof.

(a) A consists of all points of the form

$$\frac{1}{n} \text{ and } 1 + \frac{1}{n}, \quad n = 1, 2, \ldots;$$

i.e., A is the *sequence*

$$\left\{1, 2, \frac{1}{2}, 1\frac{1}{2}, \ldots, \frac{1}{n}, 1 + \frac{1}{n}, \ldots\right\}.$$

Does it converge?

(b) A is the set of all *rationals* in $(0, 1)$. Answer: $A' = [0, 1]$. Why?

(c) A is the union of the intervals

$$\left[\frac{2n}{2n+1}, \frac{2n+1}{2n+2}\right], \quad n = 0, 1, 2, \ldots.$$

(d) A consists of all points of the form

$$2^{-n} \text{ and } 2^{-n} + 2^{-n-k}, \quad n, k \in N.$$

11. Can a *sequence* $\{x_m\} \subseteq E^1$ cluster at *each* $p \in E^1$?
[Hint: See Example (e).]

12. Prove that if

$$p = \sup A \text{ or } p = \inf A \text{ in } E^1$$

$(\emptyset \neq A \subseteq E^1)$, and if $p \notin A$, then p is a cluster point of A.
[Hint: Take $G_p(\varepsilon) = (p - \varepsilon, p + \varepsilon)$. Use Theorem 2 of Chapter 2, §§8–9.]

13. Prove that a set $A \subseteq (S, \rho)$ clusters at p iff every *neighborhood* of p (see §12, Definition 1) contains infinitely many points of A; similarly for sequences. How about convergence? State it in terms of *cubic* neighborhoods in E^n.

14. Discuss Example (h) for nondegenerate intervals *in* E^n. Give a proof.

15. Prove that a set $A \neq \emptyset$ clusters at p $(p \notin A)$ iff $\rho(p, A) = 0$. (See §13, Note 3.)

16. Show that in E^n (*and in any other *normed* space $\neq \{\bar{0}\}$), the cluster points of any globe $G_{\bar{p}}(\varepsilon)$ form exactly the *closed* globe $\overline{G}_{\bar{p}}(\varepsilon)$, and that

$\overline{G}_{\bar{p}}(\varepsilon)$ is perfect. Is this true in other spaces? (Consider a *discrete* space!)

[Hint: Given $\bar{q} \in \overline{G}_{\bar{p}}(\varepsilon)$ in E^n, show that any $G_{\bar{q}}(\delta)$ overlaps with the line $\overline{pq}$. Show also that no point *outside* $\overline{G}_{\bar{q}}(\varepsilon)$ is a cluster point of $G_{\bar{p}}(\varepsilon)$.]

17. (Cantor's set.) Remove from [0, 1] the open middle third

$$\left(\frac{1}{3}, \frac{2}{3}\right).$$

From the remaining closed intervals

$$\left[0, \frac{1}{3}\right] \text{ and } \left[\frac{2}{3}, 1\right],$$

remove their open middles,

$$\left(\frac{1}{9}, \frac{2}{9}\right) \text{ and } \left(\frac{7}{9}, \frac{8}{9}\right).$$

Do the same with the remaining four closed intervals, and so on, ad infinitum. The set P which remains after *all* these (infinitely many) removals is called *Cantor's set*.

Show that P is perfect.

[Hint: If $p \notin P$, then either p is in one of the *removed* open intervals, or $p \notin [0, 1]$. In both cases, p is no cluster point of P. (Why?) Thus no p *outside* P is a cluster point.

On the other hand, if $p \in P$, show that any $G_p(\varepsilon)$ contains infinitely many *endpoints* of removed open intervals, all in P; thus $p \in P'$. Deduce that $P = P'$.]

§15. Operations on Convergent Sequences[1]

Sequences in E^1 and C can be added and multiplied *termwise*; for example, adding $\{x_m\}$ and $\{y_m\}$, one obtains the sequence with general term $x_m + y_m$. This leads to important theorems, valid also for E^n (*and other normed spaces). Theorem 1 below states, roughly, that *the limit of the sum* $\{x_m + y_m\}$ *equals the sum of* $\lim x_m$ *and* $\lim y_m$ (*if these exist*), and similarly for products and quotients (when they are defined).[2]

Theorem 1. *Let* $x_m \to q$, $y_m \to r$, *and* $a_m \to a$ *in* E^1 *or* C (*the complex field*). *Then*

(i) $x_m \pm y_m \to q \pm r$;

[1] This section (and the rest of this chapter) may be deferred until Chapter 4, §2. Then Theorems 1 and 2 may be combined with the more general theorems of Chapter 4, §3. (It is rather a matter of taste which to do *first*.)

[2] Theorem 1 is known as "*continuity of addition, multiplication, and division*" (for reasons to be clarified later). Note the restriction $a \neq 0$ in (iii).

(ii) $a_m x_m \to aq$;

(iii) $\dfrac{x_m}{a_m} \to \dfrac{q}{a}$ *if* $a \neq 0$ *and for all* $m \geq 1$, $a_m \neq 0$.

This also holds if the x_m, y_m, q, *and* r *are vectors in* E^n (**or in another normed space*), *while the* a_m *and* a *are scalars for that space.*

Proof. (i) By formula (2) of §14, we must show that

$$(\forall \varepsilon > 0)\ (\exists k)\ (\forall m > k) \quad |x_m \pm y_m - (q \pm r)| < \varepsilon.$$

Thus we fix an arbitrary $\varepsilon > 0$ and look for a suitable k. Since $x_m \to q$ and $y_m \to r$, there are k' and k'' such that

$$(\forall m > k') \quad |x_m - q| < \frac{\varepsilon}{2}$$

and

$$(\forall m > k'') \quad |y_m - r| < \frac{\varepsilon}{2}$$

(as ε is *arbitrary*, we may as well replace it by $\frac{1}{2}\varepsilon$). Then *both* inequalities hold for $m > k$, $k = \max(k', k'')$. Adding them, we obtain

$$(\forall m > k) \quad |x_m - q| + |y_m - r| < \varepsilon.$$

Hence by the triangle law,

$$|x_m - q \pm (y_m - r)| < \varepsilon, \text{ i.e., } |x_m \pm y_m - (q \pm r)| < \varepsilon \text{ for } m > k,$$

as required. □

This proof of (i) applies to sequences of vectors as well, without any change. The proof of (ii) and (iii) is sketched in Problems 1–4 below.

Note 1. By induction, parts (i) and (ii) hold for sums and products of any *finite* (but fixed) number of suitable convergent sequences.

Note 2. The theorem does not apply to *infinite* limits q, r, a.

Note 3. The assumption $a \neq 0$ in Theorem 1(iii) is important. It ensures not only that q/a is defined but also that *at most finitely many* a_m *can vanish* (see Problem 3). Since we may safely drop a finite number of terms (see Note 2 in §14), we can achieve that *no* a_m is 0, so that x_m/a_m is *defined*. It is with this understanding that part (iii) of the theorem has been formulated. The next two theorems are actually special cases of more general propositions to be proved in Chapter 4, §§3 and 5. Therefore, we only state them here, leaving the proofs as exercises, with some hints provided.

Theorem 2 (componentwise convergence)**.** *We have* $\bar{x}_m \to \bar{p}$ *in* E^n (***C^n) *iff each of the* n *components of* $\bar{x}_m$ *tends to the corresponding component of* $\bar{p}$, *i.e., iff* $x_{mk} \to p_k$, $k = 1, 2, \ldots, n$, *in* $E^1(C)$. (*See Problem* 8 *for hints.*)

Theorem 3. *Every monotone sequence* $\{x_n\} \subseteq E^*$ *has a finite or infinite limit, which equals* $\sup_n x_n$ *if* $\{x_n\}\uparrow$ *and* $\inf_n x_n$ *if* $\{x_n\}\downarrow$. *If* $\{x_n\}$ *is monotone and bounded in* E^1, *its limit is finite* (*by* Corollary 1 *of Chapter* 2, §13).

The proof was requested in Problem 9 of Chapter 2, §13. See also Chapter 4, §5, Theorem 1. An important application is the following.

Example (the number e).

Let $x_n = \left(1 + \frac{1}{n}\right)^n$ in E^1. By the binomial theorem,

$$\begin{aligned} x_n &= 1 + 1 + \frac{n(n-1)}{2!\,n^2} + \frac{n(n-1)(n-2)}{3!\,n^3} + \cdots \\ &\qquad + \frac{n(n-1)\cdots(n-(n-1))}{n!\,n^n} \\ &= 2 + \left(1 - \frac{1}{n}\right)\frac{1}{2!} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{1}{3!} + \cdots \\ &\qquad + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)\frac{1}{n!}. \end{aligned}$$

If n is replaced by $n+1$, all terms in this expansion *increase*, as does their number. Thus $x_n < x_{n+1}$, i.e., $\{x_n\}\uparrow$. Moreover, for $n > 1$,

$$\begin{aligned} 2 < x_n &< 2 + \frac{1}{2!} + \cdots + \frac{1}{n!} \le 2 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} \\ &= 2 + \frac{1}{2}\left(1 + \cdots + \frac{1}{2^{n-2}}\right) = 2 + \frac{1}{2}\,\frac{1 - \left(\frac{1}{2}\right)^{n-1}}{\frac{1}{2}} < 2 + 1 = 3. \end{aligned}$$

Thus $2 < x_n < 3$ for $n > 1$. Hence $2 < \sup_n x_n \le 3$; and by Theorem 3, $\sup_n x_n = \lim x_n$. This limit, denoted by e, plays an important role in analysis. It can be shown that it is irrational, and (to within 10^{-20}) $e = 2.71828182845904523536\ldots$. In any case,

$$2 < e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \le 3. \tag{1}$$

The following corollaries are left as exercises for the reader.

Corollary 1. *Suppose* $\lim x_m = p$ *and* $\lim y_m = q$ *exist in* E^*.

(a) *If* $p > q$, *then* $x_m > y_m$ *for all but finitely many* m.

(b) *If* $x_m \le y_m$ *for infinitely many* m, *then* $p \le q$; *i.e.*, $\lim x_m \le \lim y_m$.

This is known as *passage to the limit in inequalities*. *Caution*: The *strict* inequalities $x_m < y_m$ do not imply $p < q$ but only $p \le q$. For example, let

$$x_m = \frac{1}{m} \text{ and } y_m = 0.$$

Then

$$(\forall\, m) \quad x_m > y_m;$$

yet $\lim x_m = \lim y_m = 0$.

Corollary 2. *Let $x_m \to p$ in E^*, and let $c \in E^*$ (finite or not). Then the following are true:*

(a) *If $p > c$ (respectively, $p < c$), we have $x_m > c$ ($x_m < c$) for all but finitely many m.*

(b) *If $x_m \leq c$ (respectively, $x_m \geq c$) for infinitely many m, then $p \leq c$ ($p \geq c$).*

One can prove this from Corollary 1, with $y_m = c$ (or $x_m = c$) for all m.

Corollary 3 (rule of intermediate sequence)**.** *If $x_m \to p$ and $y_m \to p$ in E^* and if $x_m \leq z_m \leq y_m$ for all but finitely many m, then also $z_m \to p$.*

Theorem 4 (continuity of the distance function)**.** *If*

$$x_m \to p \text{ and } y_m \to q \text{ in a metric space } (S,\, \rho),$$

then

$$\rho(x_m,\, y_m) \to \rho(p,\, q) \text{ in } E^1.$$

Hint: Show that

$$|\rho(x_m,\, y_m) - \rho(p,\, q)| \leq \rho(x_m,\, p) + \rho(q,\, y_m) \to 0$$

by Theorem 1.

Problems on Limits of Sequences

See also Chapter 2, §13.

1. Prove that if $x_m \to 0$ and if $\{a_m\}$ is bounded in E^1 or C, then

$$a_m x_m \to 0.$$

This is true also if the x_m are vectors and the a_m are scalars (or vice versa).

[Hint: If $\{a_m\}$ is bounded, there is a $K \in E^1$ such that

$$(\forall\, m) \quad |a_m| < K.$$

As $x_m \to 0$,

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, m > k) \quad |x_m| < \frac{\varepsilon}{K} \text{ (why?)},$$

so $|a_m x_m| < \varepsilon$.]

2. Prove Theorem 1(ii).

[Hint: By Corollary 2(ii)(iii) in §14, we must show that $a_m x_m - aw \to 0$. Now

$$a_m x_m - aq = a_m(x_m - q) + (a_m - a)q,$$

where $x_m - q \to 0$ and $a_m - a \to 0$ by Corollary 2 of §14. Hence by Problem 1,

$$a_m(x_m - q) \to 0 \text{ and } (a_m - a)q \to 0$$

(treat q as a *constant* sequence and use Corollary 5 in §14). Now apply Theorem 1(i).]

3. Prove that if $a_m \to a$ and $a \neq 0$ in E^1 or C, then

$$(\exists\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, m > k) \quad |a_m| \geq \varepsilon.$$

(We briefly say that the a_m are *bounded away* from 0, for $m > k$.) Hence prove the *boundedness of* $\{\frac{1}{a_m}\}$ for $m > k$.

[Hint: For the first part, proceed as in the proof of Corollary 1 in §14, with $x_m = a_m$, $p = a$, and $q = 0$.

For the second part, the inequalities

$$(\forall\, m > k) \quad \left|\frac{1}{a_m}\right| \leq \frac{1}{\varepsilon}$$

lead to the desired result.]

4. Prove that if $a_m \to a \neq 0$ in E^1 or C, then

$$\frac{1}{a_m} \to \frac{1}{a}.$$

Use this and Theorem 1(ii) to prove Theorem 1(iii), noting that

$$\frac{x_m}{a_m} = x_m \cdot \frac{1}{a_m}.$$

[Hint: Use Note 3 and Problem 3 to find that

$$(\forall\, m > k) \quad \left|\frac{1}{a_m} - \frac{1}{a}\right| = \frac{1}{|a|}\ |a_m - a|\ \frac{1}{|a_m|},$$

where $\left\{\frac{1}{a_m}\right\}$ is *bounded* and $\frac{1}{|a|}\ |a_m - a| \to 0$. (Why?)

Hence, by Problem 1, $\left|\frac{1}{a_m} - \frac{1}{a}\right| \to 0$. Proceed.]

5. Prove Corollaries 1 and 2 in two ways:

(i) Use Definition 2 of Chapter 2, §13 for Corollary 1(a), treating infinite limits *separately*; then prove (b) by assuming the opposite and exhibiting a contradiction to (a).

(ii) Prove (b) *first* by using Corollary 2 and Theorem 3 of Chapter 2, §13; *then* deduce (a) by contradiction.

6. Prove Corollary 3 in two ways (cf. Problem 5).

7. Prove Theorem 4 as suggested, and also without using Theorem 1(i).

8. Prove Theorem 2.

[Hint: If $\bar{x}_m \to \bar{p}$, then

$$(\forall\, \varepsilon > 0)\ (\exists\, q)\ (\forall\, m > q) \quad \varepsilon > |\bar{x}_m - \bar{p}| \geq |x_{mk} - p_k|. \quad \text{(Why?)}$$

Thus by definition $x_{mk} \to p_k$, $k = 1, 2, \ldots, n$.

Conversely, if so, use Theorem 1(i)(ii) to obtain

$$\sum_{k=1}^{n} x_{mk}\vec{e}_k \to \sum_{k=1}^{n} p_k\vec{e}_k,$$

with $\vec{e}_k$ as in Theorem 2 of §§1–3].

8′. In Problem 8, prove the converse part *from definitions*. (Fix $\varepsilon > 0$, etc.)

9. Find the following limits in E^1, in two ways: (i) using Theorem 1, justifying each step; (ii) using definitions only.

$$\text{(a)} \lim_{m\to\infty} \frac{m+1}{m}; \qquad \text{(b)} \lim_{m\to\infty} \frac{3m+2}{2m-1};$$

$$\text{(c)} \lim_{n\to\infty} \frac{1}{1+n^2}; \qquad \text{(d)} \lim_{n\to\infty} \frac{n(n-1)}{1-2n^2}.$$

[Solution of (a) by the first method: Treat

$$\frac{m+1}{m} = 1 + \frac{1}{m}$$

as the sum of $x_m = 1$ (constant) and

$$y_m = \frac{1}{m} \to 0 \text{ (proved in §14).}$$

Thus by Theorem 1(i),

$$\frac{m+1}{m} = x_m + y_m \to 1 + 0 = 1.$$

Second method: Fix $\varepsilon > 0$ and find k such that

$$(\forall\, m > k) \quad \left|\frac{m+1}{m} - 1\right| < \varepsilon.$$

Solving for m, show that this holds if $m > \frac{1}{\varepsilon}$. Thus take an integer $k > \frac{1}{\varepsilon}$, so

$$(\forall\, m > k) \quad \left|\frac{m+1}{m} - 1\right| < \varepsilon.$$

Caution: One cannot apply Theorem 1(iii) *directly*, treating $(m+1)/m$ as the quotient of $x_m = m+1$ and $a_m = m$, because x_m and a_m *diverge* in E^1. (Theorem 1 does not apply to infinite limits.) As a remedy, we first divide the numerator and denominator by a suitable power of m (or n).]

10. Prove that

$$|x_m| \to +\infty \text{ in } E^* \text{ iff } \frac{1}{x_m} \to 0 \quad (x_m \neq 0).$$

11. Prove that if

$$x_m \to +\infty \text{ and } y_m \to q \neq -\infty \text{ in } E^*,$$

then

$$x_m + y_m \to +\infty.$$

This is written symbolically as

$$\text{“}{+\infty} + q = +\infty \text{ if } q \neq -\infty.\text{”}$$

Do also

$$\text{“}{-\infty} + q = -\infty \text{ if } q \neq +\infty.\text{”}$$

Prove similarly that

$$\text{“}(+\infty) \cdot q = +\infty \text{ if } q > 0\text{”}$$

and

$$\text{“}(+\infty) \cdot q = -\infty \text{ if } q < 0.\text{”}$$

[Hint: Treat the cases $q \in E^1$, $q = +\infty$, and $q = -\infty$ separately. Use definitions.]

12. Find the limit (or $\underline{\lim}$ and $\overline{\lim}$) of the following sequences in E^*:

(a) $x_n = 2 \cdot 4 \cdots 2n = 2^n n!$;

(b) $x_n = 5n - n^3$;

(c) $x_n = 2n^4 - n^3 - 3n^2 - 1$;

(d) $x_n = (-1)^n n!$;

(e) $x_n = \dfrac{(-1)^n}{n!}$.

[Hint for (b): $x_n = n(5 - n^2)$; use Problem 11.]

13. Use Corollary 4 in §14, to find the following:

(a) $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{1+n^2}$;

(b) $\displaystyle\lim_{n\to\infty} \frac{1-n+(-1)^n}{2n+1}$.

14. Find the following.

(a) $\displaystyle\lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2}$;

(b) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3+1}$;

(c) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^3}{n^4-1}$.

[Hint: Compute $\sum_{k=1}^{n} k^m$ using Problem 10 of Chapter 2, §§5–6.]

What is wrong with the following "solution" of (a): $\dfrac{1}{n^2} \to 0$, $\dfrac{2}{n^2} \to 0$, etc.; hence the limit is 0?

15. For each integer $m \geq 0$, let

$$S_{mn} = 1^m + 2^m + \cdots + n^m.$$

Prove by induction on m that

$$\lim_{n\to\infty} \frac{S_{mn}}{(n+1)^{m+1}} = \frac{1}{m+1}.$$

[Hint: First prove that

$$(m+1)S_{mn} = (n+1)^{m+1} - 1 - \sum_{i=0}^{m-1} \binom{m+1}{i} S_{mi}$$

by adding up the binomial expansions of $(k+1)^{m+1}$, $k = 1, \ldots, n$.]

16. Prove that

$$\lim_{n\to\infty} q^n = +\infty \text{ if } q > 1; \quad \lim_{n\to\infty} q^n = 0 \text{ if } |q| < 1; \quad \lim_{n\to\infty} 1^n = 1.$$

[Hint: If $q > 1$, put $q = 1 + d$, $d > 0$. By the binomial expansion,

$$q^n = (1+d)^n = 1 + nd + \cdots + d^n > nd \to +\infty. \quad \text{(Why?)}$$

If $|q| < 1$, then $\left|\frac{1}{q}\right| > 1$; so $\lim\left|\frac{1}{q}\right|^n = +\infty$; use Problem 10.]

17. Prove that

$$\lim_{n\to\infty} \frac{n}{q^n} = 0 \text{ if } |q| > 1, \text{ and } \lim_{n\to\infty} \frac{n}{q^n} = +\infty \text{ if } 0 < q < 1.$$

[Hint: If $|q| > 1$, use the binomial as in Problem 16 to obtain

$$|q|^n > \frac{1}{2}n(n-1)d^2, \ n \geq 2, \ \text{so } \frac{n}{|q|^n} < \frac{2}{(n-1)d^2} \to 0.$$

Use Corollary 3 with

$$x_n = 0, \ |z_n| = \frac{n}{|q|^n}, \text{ and } y_n = \frac{2}{(n-1)d^2}$$

to get $|z_n| \to 0$; hence also $z_n \to 0$ by Corollary 2(iii) of §14. In case $0 < q < 1$, use 10.]

18. Let $r, a \in E^1$. Prove that

$$\lim_{n\to\infty} n^r a^{-n} = 0 \text{ if } |a| > 1.$$

[Hint: If $r > 1$ and $a > 1$, use Problem 17 with $q = a^{1/r}$ to get $na^{-n/r} \to 0$. As

$$0 < n^r a^{-n} = (na^{-n/r})^r \leq na^{-n/r} \to 0,$$

obtain $n^r a^{-n} \to 0$.

If $r < 1$, then $n^r a^{-n} < na^{-n} \to 0$. What if $a < -1$?]

19. (Geometric series.) Prove that if $|q| < 1$, then

$$\lim_{n\to\infty}(a + aq + \cdots + aq^{n-1}) = \frac{a}{1-q}.$$

[Hint:

$$a(1 + q + \cdots + q^{n-1}) = a\frac{1-q^n}{1-q},$$

where $q^n \to 0$, by Problem 16.]

20. Let $0 < c < +\infty$. Prove that

$$\lim_{n\to\infty} \sqrt[n]{c} = 1.$$

[Hint: If $c > 1$, put $\sqrt[n]{c} = 1 + d_n$, $d_n > 0$. Expand $c = (1 + d_n)^n$ to show that

$$0 < d_n < \frac{c}{n} \to 0,$$

so $d_n \to 0$ by Corollary 3.]

21. Investigate the following sequences for monotonicity, $\underline{\lim}$, $\overline{\lim}$, and lim. (In each case, find suitable formula, or formulas, for the general term.)

(a) $2, 5, 10, 17, 26, \ldots;$

(b) $2, -2, 2, -2, \ldots;$

(c) $2, -2, -6, -10, -14, \ldots;$

(d) $1, 1, -1, -1, 1, 1, -1, -1, \ldots;$

(e) $\dfrac{3\cdot 2}{1}, \dfrac{4\cdot 6}{4}, \dfrac{5\cdot 10}{9}, \dfrac{6\cdot 14}{16}, \ldots.$

22. Do Problem 21 for the following sequences.

(a) $\dfrac{1}{2\cdot 3}, \dfrac{-8}{3\cdot 4}, \dfrac{27}{4\cdot 5}, \dfrac{-64}{5\cdot 6}, \dfrac{125}{6\cdot 7}, \ldots;$

(b) $\dfrac{2}{9}, -\dfrac{5}{9}, \dfrac{8}{9}, -\dfrac{13}{9}, \ldots;$

(c) $\dfrac{2}{3}, -\dfrac{2}{5}, \dfrac{4}{7}, -\dfrac{4}{9}, \dfrac{6}{11}, -\dfrac{6}{13}, \ldots;$

(d) $1, 3, 5, 1, 1, 3, 5, 2, 1, 3, 5, 3, \ldots, 1, 3, 5, n, \ldots;$

(e) $0.9, 0.99, 0.999, \ldots;$

(f) $+\infty, 1, +\infty, 2, +\infty, 3, \ldots;$

(g) $-\infty, 1, -\infty, \dfrac{1}{2}, \ldots, -\infty, \dfrac{1}{n}, \ldots.$

23. Do Problem 20 as follows: If $c \geq 1$, $\{\sqrt[n]{c}\}\downarrow$. (Why?) By Theorem 3, $p = \lim_{n\to\infty} \sqrt[n]{c}$ *exists* and

$$(\forall\, n) \quad 1 \leq p \leq \sqrt[n]{c}, \text{ i.e., } 1 \leq p^n \leq c.$$

By Problem 16, p cannot be > 1, so $p = 1$.

In case $0 < c < 1$, consider $\sqrt[n]{1/c}$ and use Theorem 1(iii).

24. Prove the existence of $\lim x_n$ and find it when x_n is defined inductively by

(i) $x_1 = \sqrt{2}\,,\ x_{n+1} = \sqrt{2x_n}\,;$

(ii) $x_1 = c > 0,\ x_{n+1} = \sqrt{c^2 + x_n}\,;$

(iii) $x_1 = c > 0,\ x_{n+1} = \dfrac{cx_n}{n+1}$; hence deduce that $\displaystyle\lim_{n\to\infty} \frac{c^n}{n!} = 0$.

[Hint: Show that the sequences are monotone and bounded in E^1 (Theorem 3). For example, in (ii) induction yields

$$x_n < x_{n+1} < c + 1. \quad \text{(Verify!)}$$

Thus $\lim x_n = \lim x_{n+1} = p$ *exists*. To find p, square the equation

$$x_{n+1} = \sqrt{c^2 + x_n} \quad \text{(given)}$$

and use Theorem 1 to get

$$p^2 = c^2 + p. \quad \text{(Why?)}$$

Solving for p (noting that $p > 0$), obtain

$$p = \lim x_n = \frac{1}{2}(1 + \sqrt{4c^2 + 1});$$

similarly in cases (i) and (iii).]

25. Find $\lim x_n$ in E^1 or E^* (if any), given that

(a) $x_n = (n+1)^q - n^q,\ 0 < q < 1;$

(b) $x_n = \sqrt{n}\,(\sqrt{n+1} - \sqrt{n});$

(c) $x_n = \dfrac{1}{\sqrt{n^2 + k}}\,;$

(d) $x_n = n(n+1)c^n$, with $|c| < 1$;

(e) $x_n = \sqrt[n]{\displaystyle\sum_{k=1}^{m} a_k^n}\,$, with $a_k > 0$;

(f) $x_n = \dfrac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$.

[Hints:

(a) $0 < x_n = n^q\left[\left(1 + \dfrac{1}{n}\right)^q - 1\right] < n^q\left(1 + \dfrac{1}{n} - 1\right) = n^{q-1} \to 0$. (Why?)

(b) $x_n = \dfrac{1}{1 + \sqrt{1 + 1/n}}$, where $1 < \sqrt{1 + \dfrac{1}{n}} < 1 + \dfrac{1}{n} \to 1$, so $x_n \to \frac{1}{2}$. (Why?)

(c) Verify that

$$\frac{n}{\sqrt{n^2 + n}} \le x_n \le \frac{n}{\sqrt{n^2 + 1}}\,,$$

so $x_n \to 1$ by Corollary 3. (Give a proof.)

(d) See Problems 17 and 18.

(e) Let $a = \max(a_1, \ldots, a_m)$. Prove that $a \le x_n \le a\sqrt[n]{m}$. Use Problem 20.]

The following are some harder but useful problems of theoretical importance. The explicit hints should make them not too hard.

26. Let $\{x_n\} \subseteq E^1$. Prove that if $x_n \to p$ in E^1, then also

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = p$$

(i.e., p is also the limit of the sequence of the *arithmetic means* of the x_n).

[Solution: Fix $\varepsilon > 0$. Then

$$(\exists\, k)\ (\forall\, n > k) \quad p - \frac{\varepsilon}{4} < x_n < p + \frac{\varepsilon}{4}.$$

Adding $n - k$ inequalities, get

$$(n-k)\Big(p - \frac{\varepsilon}{4}\Big) < \sum_{i=k+1}^{n} x_i < (n-k)\Big(p + \frac{\varepsilon}{4}\Big).$$

With k so fixed, we thus have

$$(\forall\, n > k) \quad \frac{n-k}{n}\Big(p - \frac{\varepsilon}{4}\Big) < \frac{1}{n}\ (x_{k+1} + \cdots + x_n) < \frac{n-k}{n}\Big(p + \frac{\varepsilon}{4}\Big). \tag{i}$$

Here, with k and ε fixed,

$$\lim_{n\to\infty} \frac{n-k}{n}\Big(p - \frac{\varepsilon}{4}\Big) = p - \frac{\varepsilon}{4}.$$

Hence, as $p - \frac{1}{2}\varepsilon < p - \frac{1}{4}\varepsilon$, there is k' such that

$$(\forall\, n > k') \quad p - \frac{\varepsilon}{2} < \frac{n-k}{n}\Big(p - \frac{\varepsilon}{4}\Big).$$

Similarly,

$$(\exists\, k'')\ (\forall\, n > k'') \quad \frac{n-k}{n}\Big(p + \frac{\varepsilon}{4}\Big) < p + \frac{\varepsilon}{2}.$$

Combining this with (i), we have, for $K' = \max(k, k', k'')$,

$$(\forall\, n > K') \quad p - \frac{\varepsilon}{2} < \frac{1}{n}(x_{k+1} + \cdots + x_n) < p + \frac{\varepsilon}{2}. \tag{ii}$$

Now with k fixed,

$$\lim_{n\to\infty} \frac{1}{n}(x_1 + x_2 + \cdots + x_k) = 0.$$

Hence

$$(\exists\, K'')\ (\forall\, n > K'') \quad -\frac{\varepsilon}{2} < \frac{1}{n}(x_1 + \cdots + x_k) < \frac{\varepsilon}{2}.$$

Let $K = \max(K', K'')$. Then combining with (ii), we have

$$(\forall n > K) \quad p - \varepsilon < \frac{1}{n}(x_1 + \cdots + x_n) < p + \varepsilon,$$

and the result follows.]

26′ Show that the result of Problem 26 holds also for infinite limits $p = \pm\infty \in E^*$.

27. Prove that if $x_n \to p$ in E^* $(x_n > 0)$, then

$$\lim_{n\to\infty} \sqrt[n]{x_1 x_2 \cdots x_n} = p.$$

[Hint: Let first $0 < p < +\infty$. Given $\varepsilon > 0$, use density to fix $\delta > 1$ so close to 1 that

$$p - \varepsilon < \frac{p}{\delta} < p < p\delta < p + \varepsilon.$$

As $x_n \to p$,

$$(\exists\, k)\ (\forall\, n > k) \quad \frac{p}{\sqrt[4]{\delta}} < x_n < p\sqrt[4]{\delta}.$$

Continue as in Problem 26, replacing ε by δ, and multiplication by addition (also subtraction by division, etc., as shown above).[3] Find a similar solution for the case $p = +\infty$. Note the result of Problem 20.]

28. Disprove by counterexamples the converse implications in Problems 26 and 27. For example, consider the sequences

$$1,\ -1,\ 1,\ -1,\ \ldots$$

and

$$\frac{1}{2},\ 2,\ \frac{1}{2},\ 2,\ \frac{1}{2},\ 2,\ \ldots.$$

29. Prove the following.

(i) If $\{x_n\} \subset E^1$ and $\lim_{n\to\infty}(x_{n+1} - x_n) = p$ in E^*, then $\frac{x_n}{n} \to p$.

(ii) If $\{x_n\} \subset E^1$ $(x_n > 0)$ and if $\frac{x_{n+1}}{x_n} \to p \in E^*$, then $\sqrt[n]{x_n} \to p$.

Disprove the converse statements by counterexamples.

[Hint: For (i), let $y_1 = x_1$ and $y_n = x_n - x_{n-1}$, $n = 2, 3, \ldots$. Then $y_n \to p$ and

$$\frac{1}{n}\sum_{i=1}^{n} y_i = \frac{x_n}{n},$$

so Problems 26 and 26′ apply.

For (ii), use Problem 27. See Problem 28 for examples.]

30. From Problem 29 deduce that

(a) $\lim_{n\to\infty} \sqrt[n]{n!} = +\infty$;

[3] Another solution (reducing all to Problem 26) will be obtained by applying logarithms.

(b) $\lim_{n\to\infty} \frac{n+1}{n!} = 0;$

(c) $\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}} = e;$

(d) $\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{n!} = \frac{1}{e};$

(e) $\lim_{n\to\infty} \sqrt[n]{n} = 1.$

31. Prove that

$$\lim_{n\to\infty} x_n = \frac{a+2b}{3},$$

given

$$x_0 = a,\ x_1 = b,\ \text{and}\ x_{n+2} = \frac{1}{2}(x_n + x_{n+1}).$$

[Hint: Show that the differences $d_n = x_n - x_{n-1}$ form a geometric sequence, with ratio $q = -\frac{1}{2}$, and $x_n = a + \sum_{k=1}^{n} d_k$. Then use the result of Problem 19.]

⇒**32.** For any sequence $\{x_n\} \subseteq E^1$, prove that

$$\underline{\lim}\, x_n \le \underline{\lim}\, \frac{1}{n} \sum_{i=1}^{n} x_i \le \overline{\lim}\, \frac{1}{n} \sum_{i=1}^{n} x_i \le \overline{\lim}\, x_n.$$

Hence find a new solution of Problems 26 and 26′.

[Proof for $\overline{\lim}$: Fix any $k \in N$. Put

$$c = \sum_{i=1}^{k} x_i \text{ and } b = \sup_{i\ge k} x_i.$$

Verify that

$$(\forall\, n > k) \quad x_{k+1} + x_{k+2} + \cdots + x_n \le (n-k)b.$$

Add c on both sides and divide by n to get

$$(\forall\, n > k) \quad \frac{1}{n} \sum_{i=1}^{n} x_i \le \frac{c}{n} + \frac{n-k}{n} b. \tag{i*}$$

Now fix any $\varepsilon > 0$, and first let $|b| < +\infty$. As $\frac{c}{n} \to 0$ and $\frac{n-k}{n} b \to b$, there is $n_k > k$ such that

$$(\forall\, n > n_k) \quad \frac{c}{n} < \frac{\varepsilon}{2} \text{ and } \frac{n-k}{n} b < b + \frac{\varepsilon}{2}.$$

Thus by (i*),

$$(\forall\, n > n_k) \quad \frac{1}{n} \sum_{i=1}^{n} x_i \le \varepsilon + b.$$

This clearly holds also if $b = \sup_{i\ge k} x_i = +\infty$. Hence also

$$\sup_{n\ge n_k} \frac{1}{n} \sum_{i=1}^{n} x_i \le \varepsilon + \sup_{i\ge k} x_i.$$

As k and ε were *arbitrary*, we may let first $k \to +\infty$, then $\varepsilon \to 0$, to obtain

$$\overline{\lim}\, \frac{1}{n} \sum_{i=1}^{n} x_i \leq \lim_{k\to\infty} \sup_{i\geq k} x_i = \overline{\lim}\, x_n. \quad \text{(Explain!)]}$$

⇒**33.** Given $\{x_n\} \subseteq E^1$, $x_n > 0$, prove that

$$\underline{\lim}\, x_n \leq \underline{\lim}\, \sqrt[n]{x_1 x_2 \cdots x_n} \text{ and } \overline{\lim}\, \sqrt[n]{x_1 x_2 \cdots x_n} \leq \overline{\lim}\, x_n.$$

Hence obtain a new solution for Problem 27.

[Hint: Proceed as suggested in Problem 32, replacing addition by multiplication.]

34. Given $x_n, y_n \in E^1$ $(y_n > 0)$, with

$$x_n \to p \in E^* \text{ and } b_n = \sum_{i=1}^{n} y_i \to +\infty,$$

prove that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} y_i} = p.$$

Note that Problem 26 is a special case of Problem 34 (take all $y_n = 1$).

[Hint for a *finite* p: Proceed as in Problem 26. However, before adding the $n - k$ inequalities, multiply by y_i and obtain

$$\left(p - \frac{\varepsilon}{4}\right) \sum_{i=k+1}^{n} y_i < \sum_{i=k+1}^{n} x_i y_i < \left(p + \frac{\varepsilon}{4}\right) \sum_{i=k+1}^{n} y_i.$$

Put $b_n = \sum_{i=1}^{n} y_i$ and show that

$$\frac{1}{b_n} \sum_{i=k+1}^{n} x_i y_i = 1 - \frac{1}{b_n} \sum_{i=1}^{k} x_i y_i,$$

where $b_n \to +\infty$ (by assumption), so

$$\frac{1}{b_n} \sum_{i=1}^{k} x_i y_i \to 0 \quad \text{(for a fixed } k\text{)}.$$

Proceed. Find a proof for $p = \pm\infty$.]

35. Do Problem 34 by considering $\underline{\lim}$ and $\overline{\lim}$ as in Problem 32.

[Hint: Replace $\dfrac{c}{n}$ by $\dfrac{c}{b_n}$, where $b_n = \sum_{i=1}^{n} y_i \to +\infty$.]

36. Prove that if $u_n, v_n \in E^1$, with $\{v_n\}\!\uparrow$ (strictly) and $v_n \to +\infty$, and if

$$\lim_{n\to\infty} \frac{u_n - u_{n-1}}{v_n - v_{n-1}} = p \quad (p \in E^*),$$

then also

$$\lim_{n\to\infty} \frac{u_n}{v_n} = p.$$

[Hint: The result of Problem 34, with

$$x_n = \frac{u_n - u_{n-1}}{v_n - v_{n-1}} \text{ and } y_n = v_n - v_{n-1}.$$

leads to the final result.]

37. From Problem 36 obtain a new solution for Problem 15. Also prove that

$$\lim_{n\to\infty}\left(\frac{S_{mn}}{n^{m+1}} - \frac{1}{m+1}\right) = \frac{1}{2}.$$

[Hint: For the first part, put

$$u_n = S_{mn} \text{ and } v_n = n^{m+1}.$$

For the second, put

$$u_n = (m+1)S_{mn} - n^{m+1} \text{ and } v_n = n^m(m+1).]$$

38. Let $0 < a < b < +\infty$. Define inductively: $a_1 = \sqrt{ab}$ and $b_1 = \frac{1}{2}(a+b)$;

$$a_{n+1} = \sqrt{a_n b_n} \text{ and } b_{n+1} = \frac{1}{2}(a_n + b_n),\ n = 1, 2, \ldots.$$

Then $a_{n+1} < b_{n+1}$ for

$$b_{n+1} - a_{n+1} = \frac{1}{2}(a_n + b_n) - \sqrt{a_n b_n} = \frac{1}{2}(\sqrt{b_n} - \sqrt{a_n})^2 > 0.$$

Deduce that

$$a < a_n < a_{n+1} < b_{n+1} < b_n < b,$$

so $\{a_n\}\uparrow$ and $\{b_n\}\downarrow$. By Theorem 3, $a_n \to p$ and $b_n \to q$ for some $p, q \in E^1$. Prove that $p = q$, i.e.,

$$\lim a_n = \lim b_n.$$

(This is *Gauss's arithmetic–geometric mean* of a and b.)
[Hint: Take limits of both sides in $b_{n+1} = \frac{1}{2}(a_n + b_n)$ to get $q = \frac{1}{2}(p+q)$.]

39. Let $0 < a < b$ in E^1. Define inductively $a_1 = a$, $b_1 = b$,

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n}, \text{ and } b_{n+1} = \frac{1}{2}(a_n + b_n), \quad n = 1, 2, \ldots.$$

Prove that

$$\sqrt{ab} = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

[Hint: Proceed as in Problem 38.]

40. Prove the *continuity of dot multiplication*, namely, if

$$\bar{x}_n \to \bar{q} \text{ and } \bar{y}_n \to \bar{r} \text{ in } E^n$$

(*or in another *Euclidean space*; see §9), then

$$\bar{x}_n \cdot \bar{y}_n \to \bar{q} \cdot \bar{r}.$$

§16. More on Cluster Points and Closed Sets. Density

I. The notions of *cluster point* and *closed set* (§§12, 14) can be characterized in terms of convergent sequences. We start with cluster points.

Theorem 1.

(i) *A sequence $\{x_m\} \subseteq (S, \rho)$ clusters at a point $p \in S$ iff it has a subsequence $\{x_{m_n}\}$ converging to p.*[1]

(ii) *A set $A \subseteq (S, \rho)$ clusters at $p \in S$ iff p is the limit of some sequence $\{x_n\}$ of points of A other than p; if so, the terms x_n can be made distinct.*

Proof. (i) If $p = \lim_{n\to\infty} x_{m_n}$, then by definition each globe about p contains all but finitely many x_{m_n}, hence *infinitely many* x_m. Thus p is a cluster point.

Conversely, if so, consider in particular the globes

$$G_p\left(\frac{1}{n}\right), \quad n = 1, 2, \ldots.$$

By assumption, $G_p(1)$ contains some x_m. Thus fix

$$x_{m_1} \in G_p(1).$$

Next, choose a term

$$x_{m_2} \in G_p\left(\frac{1}{2}\right) \text{ with } m_2 > m_1.$$

(Such terms *exist* since $G_p(\frac{1}{2})$ contains *infinitely many* x_m.) Next, fix

$$x_{m_3} \in G_p\left(\frac{1}{3}\right), \text{ with } m_3 > m_2 > m_1,$$

and so on.

Thus, step by step (inductively), select a sequence of subscripts

$$m_1 < m_2 < \cdots < m_n < \cdots$$

that determines a *subsequence* (see Chapter 1, §8) such that

$$(\forall\, n) \quad x_{m_n} \in G_p\left(\frac{1}{n}\right), \text{ i.e., } \rho(x_{m_n}, p) < \frac{1}{n} \to 0,$$

[1] Therefore, cluster points of $\{x_m\}$ are also called *subsequential limits*.

whence $\rho(x_{m_n}, p) \to 0$, or $x_{m_n} \to p$. (Why?) Thus we have found a subsequence $x_{m_n} \to p$, and assertion (i) is proved.

Assertion (ii) is proved quite similarly—proceed as in the proof of Corollary 6 in §14; the inequalities $m_1 < m_2 < \cdots$ are not needed here. □

Examples.

(a) Recall that the set R of all rationals clusters at each $p \in E^1$ (§14, Example (e)). Thus by Theorem 1(ii), *each real p is the limit of a sequence of rationals*. See also Problem 6 of §12 for *$\bar{p}$ in E^n*.

(b) The sequence

$$0,\ 1,\ 0,\ 1,\ \ldots$$

has two convergent subsequences,

$$x_{2n} = 1 \to 1 \text{ and } x_{2n-1} = 0 \to 0.$$

Thus by Theorem 1(i), it clusters at 0 and 1.

Interpret Example (f) and Problem 10(a) in §14 similarly.

As we know, even infinite sets may have no cluster points (take N in E^1). However, a *bounded* infinite set or sequence *in E^n (*or C^n) must* cluster. This important theorem (due to Bolzano and Weierstrass) is proved next.

Theorem 2 (Bolzano–Weierstrass)**.**

(i) *Each bounded infinite set or sequence A in E^n (*or C^n) has at least one cluster point $\bar{p}$ there (possibly outside A).*

(ii) *Thus each bounded sequence in E^n (*C^n) has a convergent subsequence.*

Proof. Take first a bounded sequence $\{z_m\} \subseteq [a, b]$ *in* E^1. Let

$$p = \overline{\lim}\, z_m.$$

By Theorem 2(i) of Chapter 2, §13, $\{z_m\}$ clusters at p. Moreover, as

$$a \le z_m \le b,$$

we have

$$a \le \inf z_m \le p \le \sup z_m \le b$$

by Corollary 1 of Chapter 2, §13. Thus

$$p \in [a, b] \subseteq E^1,$$

and so $\{z_m\}$ clusters *in* E^1.

Assertion (ii) now follows—*for* E^1—by Theorem 1(i) above.

Next, take

$$\{\bar{z}_m\} \subseteq E^2,\ \ \bar{z}_m = (x_m,\, y_m);\ \ x_m,\, y_m \in E^1.$$

If $\{\bar{z}_m\}$ is bounded, all $\bar{z}_m$ are in some *square* $[\bar{a}, \bar{b}]$. (Why?) Let

$$\bar{a} = (a_1, a_2) \text{ and } \bar{b} = (b_1, b_2).$$

Then

$$a_1 \leq x_m \leq b_1 \text{ and } a_2 \leq y_m \leq b_2 \textit{ in } E^1.$$

Thus by the first part of the proof, $\{x_m\}$ has a convergent subsequence

$$x_{m_k} \to p_1 \text{ for some } p_1 \in [a_1, b_1].$$

For simplicity, we henceforth write x_m for x_{m_k}, y_m for y_{m_k}, and $\bar{z}_m$ for $\bar{z}_{m_k}$. Thus $\bar{z}_m = (x_m, y_m)$ is now a *subsequence*, with $x_m \to p_1$, and $a_2 \leq y_m \leq b_2$, as before.

We now *reapply this process to* $\{y_m\}$ and obtain a subsubsequence

$$y_{m_i} \to p_2 \text{ for some } p_2 \in [a_2, b_2].$$

The corresponding terms x_{m_i} still tend to p_1 by Corollary 3 of §14. Thus we have a subsequence

$$\bar{z}_{m_i} = (x_{m_i}, y_{m_i}) \to (p_1, p_2) \quad \text{in } E^2$$

by Theorem 2 in §15. Hence $\bar{p} = (p_1, p_2)$ is a cluster point of $\{\bar{z}_m\}$. Note that $\bar{p} \in [\bar{a}, \bar{b}]$ (see above). This proves the theorem for sequences *in* E^2 (hence in C).

The proof for E^n is similar; one only has to take subsequences n *times*. (*The same applies to C^n with real components replaced by complex ones.)

Now take a bounded infinite *set* $A \subset E^n$ ($^*C^n$). Select from it an infinite sequence $\{\bar{z}_m\}$ of *distinct* points (see Chapter 1, §9, Problem 5). By what was shown above, $\{\bar{z}_m\}$ clusters at some point $\bar{p}$, so each $G_{\bar{p}}$ contains infinitely many distinct points $\bar{z}_m \in A$. Thus by definition, A clusters at $\bar{p}$. □

Note 1. We have also proved that if $\{\bar{z}_m\} \subseteq [\bar{a}, \bar{b}] \subset E^n$, then $\{\bar{z}_m\}$ has a cluster point *in* $[\bar{a}, \bar{b}]$. (This applies to *closed* intervals only.)

Note 2. The theorem may fail in spaces other than E^n ($^*C^n$). For example, in a *discrete* space, all sets are bounded, but *no* set can cluster.

II. Cluster points are closely related to the following notion.

Definition 1.

> The *closure* of a set $A \subseteq (S, \rho)$, denoted $\overline{A}$, is the union of A and the set of all cluster points of A (call it A'). Thus $\overline{A} = A \cup A'$.

Theorem 3. *We have $p \in \overline{A}$ in (S, ρ) iff each globe $G_p(\delta)$ about p meets A, i.e.,*

$$(\forall \delta > 0) \quad A \cap G_p(\delta) \neq \emptyset.$$

Equivalently, $p \in \overline{A}$ *iff*

$$p = \lim_{n\to\infty} x_n \text{ for some } \{x_n\} \subseteq A.$$

The proof is as in Corollary 6 of §14 and Theorem 1. (Here, however, the x_n *need not* be distinct or different from p.) The details are left to the reader.

This also yields the following new characterization of *closed sets* (cf. §12).

Theorem 4. *A set $A \subseteq (S, \rho)$ is closed iff one of the following conditions holds.*

(i) *A contains all its cluster points (or has none); i.e., $A \supseteq A'$.*

(ii) $A = \overline{A}$.

(iii) *A contains the limit of each convergent sequence $\{x_n\} \subseteq A$ (if any).*[2]

Proof. Parts (i) and (ii) are equivalent since

$$A \supseteq A' \iff A = A \cup A' = \overline{A}. \quad \text{(Explain!)}$$

Now let A be closed. If $p \notin A$, then $p \in -A$; therefore, by Definition 3 in §12, some G_p *fails* to meet A ($G_p \cap A = \emptyset$). Hence *no* $p \in -A$ is a cluster point, or the limit of a sequence $\{x_n\} \subseteq A$. (This would contradict Definitions 1 and 2 of §14.) Consequently, all such cluster points and limits *must* be in A, as claimed.

Conversely, suppose A is *not* closed, so $-A$ is not open. Then $-A$ has a *noninterior* point p; i.e., $p \in -A$ but *no G_p is entirely in $-A$*. This means that *each G_p meets A*. Thus

$$p \in \overline{A} \text{ (by Theorem 3)},$$

and

$$p = \lim_{n\to\infty} x_n \text{ for some } \{x_n\} \subseteq A \text{ (by the same theorem)},$$

even though $p \notin A$ (for $p \in -A$).

We see that (iii) and (ii), hence also (i), *fail* if A is not closed and hold if A *is* closed. (See the first part of the proof.) Thus the theorem is proved. □

The following corollaries are left as exercises (see Problems 6–9).

Corollary 1. $\overline{\emptyset} = \emptyset$.

Corollary 2. $A \subseteq B \implies \overline{A} \subseteq \overline{B}$.

Corollary 3. *$\overline{A}$ is always a closed set $\supseteq A$.*

[2] Property (iii) is often called the *sequential* closedness of A.

Corollary 4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$ *(the closure of* $A \cup B$ *equals the union of* $\overline{A}$ *and* $\overline{B}$*).*

III. As we know, the rationals are *dense in* E^1 (Theorem 3 of Chapter 2, §10). This means that every globe $G_p(\delta) = (p - \delta, p + \delta)$ in E^1 contains rationals. Similarly (see Problem 6 in §12), the set R^n of all rational points is dense in E^n. We now generalize this idea for arbitrary sets in a metric space (S, ρ).

Definition 2.

Given $A \subseteq B \subseteq (S, \rho)$, we say that A is *dense* in B iff each globe G_p, $p \in B$, meets A. By Theorem 3, this means that *each* $p \in B$ *is in* $\overline{A}$; i.e.,

$$p = \lim_{n \to \infty} x_n \quad \text{for some } \{x_n\} \subseteq A.$$

Equivalently, $A \subseteq B \subseteq \overline{A}$.[3]

Summing up, we have the following:

$$A \text{ is open iff } A = A^0.$$

$$A \text{ is closed iff } A = \overline{A}; \text{ equivalently, iff } A \supseteq A'.$$

$$A \text{ is dense in } B \text{ iff } A \subseteq B \subseteq \overline{A}.$$

$$A \text{ is perfect iff } A = A'.$$[4]

Problems on Cluster Points, Closed Sets, and Density

1. Complete the proof of Theorem 1(ii).

2. Prove that $\overline{R} = E^1$ and $\overline{R^n} = E^n$ (Example (a)).

3. Prove Theorem 2 for E^3. Prove it for E^n (*and C^n) *by induction on* n.

4. Verify Note 2.

5. Prove Theorem 3.

6. Prove Corollaries 1 and 2.

7. Prove that $(A \cup B)' = A' \cup B'$.
[Hint: Show by contradiction that $p \notin (A' \cup B')$ *excludes* $p \in (A \cup B)'$. Hence $(A \cup B)' \subseteq A' \cup B'$. Then show that $A' \subseteq (A \cup B)'$, etc.]

8. From Problem 7, deduce that $A \cup B$ is closed if A and B are. Then prove Corollary 4. By induction, extend both assertions to any *finite* number of sets.

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[3] If B is *closed* (e.g., if $B = S$) this means that $\overline{A} = B$. Why?

[4] See §14, the remarks following Example (i).

9. From Theorem 4, prove that if the sets A_i $(i \in I)$ are closed, so is $\bigcap_{i\in I} A_i$.

10. Prove Corollary 3 from Theorem 3. Deduce that $\overline{\overline{A}} = \overline{A}$ and prove footnote 3.
[Hint: Consider Figure 7 and Example (1) in §12 when using Theorem 3 (twice).]

11. Prove that $\overline{A}$ is contained in any closed superset of A and is the intersection of all such supersets.
[Hint: Use Corollaries 2 and 3.]

12. (i) Prove that a *bounded* sequence $\{\bar{x}_m\} \subseteq E^n$ $({}^*C^n)$ converges to $\bar{p}$ iff $\bar{p}$ is its *only* cluster point.

(ii) Disprove it for

(a) *unbounded* $\{\bar{x}_m\}$ and

(b) *other* spaces.

[Hint: For (i), if $\bar{x}_m \to \bar{p}$ *fails*, some $G_{\bar{p}}$ leaves out *infinitely* many $\bar{x}_m$. These $\bar{x}_m$ form a bounded subsequence that, by Theorem 2, clusters at some $\bar{q} \neq \bar{p}$. (Why?) Thus $\bar{q}$ is *another* cluster point (contradiction!).

For (ii), consider (a) Example (f) in §14 and (b) Problem 10 in §14, with $(0, 2]$ as a *subspace* of E^1.]

13. In each case of Problem 10 in §14, find $\overline{A}$. Is A closed? (Use Theorem 4.)

14. Prove that if $\{b_n\} \subseteq B \subseteq \overline{A}$ in (S, ρ), there is a sequence $\{a_n\} \subseteq A$ such that $\rho(a_n, b_n) \to 0$. Hence $a_n \to p$ iff $b_n \to p$.
[Hint: Choose $a_n \in G_{b_n}(1/n)$.]

15. We have, by definition,

$$p \in A^0 \text{ iff } (\exists\, \delta > 0)\ G_p(\delta) \subseteq A;$$

hence

$$p \notin A^0 \text{ iff } (\forall\, \delta > 0)\ G_p(\delta) \not\subseteq A, \text{ i.e., } G_p(\delta) - A \neq \emptyset.$$

(See Chapter 1, §§1–3.) Find such *quantifier formulas* for $p \in \overline{A}$, $p \notin \overline{A}$, $p \in A'$, and $p \notin A'$.
[Hint: Use Corollary 6 in §14, and Theorem 3 in §16.]

16. Use Problem 15 to prove that

(i) $-(\overline{A}) = (-A)^0$ and

(ii) $-(A^0) = \overline{-A}$.

17. Show that

$$\overline{A} \cap (\overline{-A}) = \text{bd}\, A \ (\textit{boundary} \text{ of } A);$$

cf. §12, Problem 18. Hence prove again that A is closed iff $A \supseteq \text{bd}\, A$.
[Hint: Use Theorem 4 and Problem 16 above.]

***18.** A set A is said to be *nowhere dense* in (S, ρ) iff $(\overline{A})^0 = \emptyset$. Show that Cantor's set P (§14, Problem 17) is nowhere dense.
[Hint: P is closed, so $\overline{P} = P$.]

***19.** Give another proof of Theorem 2 *for* E^1.
[Hint: Let $A \subseteq [a, b]$. Put

$$Q = \{x \in [a, b] \mid x \text{ exceeds infinitely many points (or terms) of } A\}.$$

Show that Q is bounded and nonempty, so it has a glb, say, $p = \inf A$. Show that A clusters *at* p.]

***20.** For any set $A \subseteq (S, \rho)$ define

$$G_A(\varepsilon) = \bigcup_{x \in A} G_x(\varepsilon).$$

Prove that

$$\overline{A} = \bigcap_{n=1}^{\infty} G_A\left(\frac{1}{n}\right).$$

***21.** Prove that

$$\overline{A} = \{x \in S \mid \rho(x, A) = 0\}; \text{ see §13, Note 3.}$$

Hence deduce that a set A in (S, ρ) is closed iff

$$(\forall\, x \in S) \quad \rho(x, A) = 0 \Longrightarrow x \in A.$$

§17. Cauchy Sequences. Completeness

A convergent sequence is characterized by the fact that its terms x_m become (and stay) arbitrarily close to its limit, as $m \to +\infty$. Due to this, however, they also get close *to each other*; in fact, $\rho(x_m, x_n)$ can be made arbitrarily small for sufficiently large m and n. It is natural to ask whether the latter property, in turn, implies the existence of a limit. This problem was first studied by Augustin-Louis Cauchy (1789–1857). Thus we shall call such sequences *Cauchy sequences*. More precisely, we formulate the following.

Definition 1.

A sequence $\{x_m\} \subseteq (S, \rho)$ is called a *Cauchy sequence* (we briefly say that "$\{x_m\}$ is Cauchy") iff, given any $\varepsilon > 0$ (no matter how small), we have $\rho(x_m, x_n) < \varepsilon$ for *all but finitely many* m *and* n. In symbols,

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, m, n > k) \quad \rho(x_m, x_n) < \varepsilon. \tag{1}$$

Observe that here we only deal with *terms* x_m, x_n, not with any other point. The limit (if any) is not involved, and *we do not have to know it in advance*. We shall now study the relationship between property (1) and convergence.

Theorem 1. *Every convergent sequence* $\{x_m\} \subseteq (S, \rho)$ *is Cauchy.*

Proof. Let $x_m \to p$. Then given $\varepsilon > 0$, there is a k such that

$$(\forall\, m > k) \quad \rho(x_m, p) < \frac{\varepsilon}{2}.$$

As this holds for *any* $m > k$, it also holds for any other term x_n with $n > k$. Thus

$$(\forall\, m, n > k) \quad \rho(x_m, p) < \frac{\varepsilon}{2} \text{ and } \rho(p, x_n) < \frac{\varepsilon}{2}.$$

Adding and using the triangle inequality, we get

$$\rho(x_m, x_n) \leq \rho(x_m, p) + \rho(p, x_n) < \varepsilon,$$

and (1) is proved. □

Theorem 2. *Every Cauchy sequence* $\{x_m\} \subseteq (S, \rho)$ *is bounded.*

Proof. We must show that all x_m are in some globe. First we try an arbitrary radius ε. Then by (1), there is k such that $\rho(x_m, x_n) < \varepsilon$ for $m, n > k$. Fix some $n > k$. Then

$$(\forall\, m > k)\ \rho(x_m, x_n) < \varepsilon, \text{ i.e., } x_m \in G_{x_n}(\varepsilon).$$

Thus the globe $G_{x_n}(\varepsilon)$ contains all x_m except possibly the k terms $x_1, \ldots, x_k$. To include them as well, we only have to take a larger radius r, greater than $\rho(x_m, x_n)$, $m = 1, \ldots, k$. Then *all* x_m are in the enlarged globe $G_{x_n}(r)$. □

Note 1. In E^1, under the standard metric, only sequences with *finite* limits are regarded as convergent. If $x_n \to \pm\infty$, then $\{x_n\}$ is not even a Cauchy sequence *in* E^1 (in view of Theorem 2); but in E^*, *under a suitable metric* (cf. Problem 5 in §11), it is convergent (hence also Cauchy and bounded).

Theorem 3. *If a Cauchy sequence* $\{x_m\}$ *clusters at a point* p*, then* $x_m \to p$*.*

Proof. We want to show that $x_m \to p$, i.e., that

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, m > k) \quad \rho(x_m, p) < \varepsilon.$$

Thus we fix $\varepsilon > 0$ and look for a suitable k. Now as $\{x_m\}$ is Cauchy, there is a k such that

$$(\forall\, m, n > k) \quad \rho(x_m, x_n) < \frac{\varepsilon}{2}.$$

Also, as p is a cluster point, the globe $G_p(\frac{\varepsilon}{2})$ contains *infinitely many* x_n, so we can fix one with $n > k$ (k as above). Then $\rho(x_n, p) < \frac{\varepsilon}{2}$ and, as noted above, also $\rho(x_m, x_n) < \frac{\varepsilon}{2}$ for $m > k$. Hence

$$(\forall\, m > k) \quad \rho(x_m, x_n) + \rho(x_n, p) < \varepsilon,$$

implying $\rho(x_m, p) \le \rho(x_m, x_n) + \rho(x_n, p) < \varepsilon$, as required. □

Note 2. It follows that a Cauchy sequence can have *at most one* cluster point p, for p is also its limit and hence *unique*; see §14, Corollary 1.

These theorems show that Cauchy sequences behave very much like convergent ones. Indeed, our next theorem (a famous result by Cauchy) shows that, *in* E^n (**and* C^n) the two kinds of sequences coincide.

Theorem 4 (Cauchy's convergence criterion)**.** *A sequence* $\{\bar{x}_m\}$ *in* E^n (**or* C^n) *converges if and only if it is a Cauchy sequence.*

Proof. If $\{x_m\}$ converges, it is Cauchy by Theorem 1.

Conversely, let $\{x_m\}$ be a Cauchy sequence. Then by Theorem 2, it is bounded. Hence by the Bolzano–Weierstrass theorem (Theorem 2 of §16), it has a cluster point $\bar{p}$. Thus by Theorem 3 above, it converges to $\bar{p}$, and all is proved. □

Unfortunately, this theorem (along with the Bolzano–Weierstrass theorem used in its proof) does not hold in *all* metric spaces. It even fails in some subspaces of E^1. For example, we have

$$x_m = \frac{1}{m} \to 0 \text{ in } E^1.$$

By Theorem 1, this sequence, being convergent, is also a Cauchy sequence. Moreover, it still preserves (1) even if we *remove* the point 0 from E^1 since the distances $\rho(x_m, x_n)$ remain the same. However, in the resulting subspace $S = E^1 - \{0\}$, the sequence no longer converges because its limit (and unique cluster point) 0 has disappeared, leaving a "gap" in its place. Thus we have a Cauchy sequence in S, *without a limit or cluster points*, so Theorem 4 fails in S (along with the Bolzano–Weierstrass theorem).

Quite similarly, both theorems fail in $(0, 1)$ (but not in $[0, 1]$) as a subspace of E^1. By analogy to incomplete ordered fields, it is natural to say that S is "incomplete" because of the missing cluster point 0, and call a space (or subspace) "complete" if it has no such "gaps," i.e., if Theorem 4 holds in it. Thus we define as follows.

Definition 2.

> A metric space (or subspace) (S, ρ) is said to be *complete* iff every Cauchy sequence in S converges to some point p *in* S.
>
> Similarly, a *set* $A \subseteq (S, \rho)$ is called *complete* iff each Cauchy sequence $\{x_m\} \subseteq A$ converges to some point p *in* A, i.e., iff (A, ρ) is complete as a metric subspace of (S, ρ).

In particular, E^n (**and* C^n) *are complete* by Theorem 4. The sets $(0, 1)$ and $E^1 - \{0\}$ are incomplete in E^1, but $[0, 1]$ is complete. Indeed, we have the following theorem.

***Theorem 5.**

(i) *Every closed set in a complete space is complete itself.*

(ii) *Every complete set* $A \subseteq (S, \rho)$ *is necessarily closed.*[1]

Proof. (i) Let A be a closed set in a complete space (S, ρ). We have to show that Theorem 4 holds in A (as it does in S). Thus we fix any Cauchy sequence $\{x_m\} \subseteq A$ and prove that it converges to some p *in* A.

Now, since S is complete, the Cauchy sequence $\{x_m\}$ has a limit p *in* S. As A is *closed*, however, that limit must be *in* A by Theorem 4 in §16. Thus (i) is proved.

(ii) Now let A be complete in a metric space (S, ρ). To prove that A is closed, we again use Theorem 4 of §16. Thus we fix any *convergent* sequence $\{x_m\} \subseteq A$, $x_m \to p \in S$, and show that p must be *in* A.

Now, since $\{x_m\}$ converges in S, it is a Cauchy sequence, in S *as well as in* A. Thus by the assumed completeness of A, it has a limit q *in* A. Then, however, the uniqueness of $\lim_{m\to\infty} x_m$ (in S) implies that $p = q \in A$, so that p is in A, indeed. $\square$

Problems on Cauchy Sequences

1. Without using Theorem 4, prove that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in E^1 (or C), so also are

$$\text{(i) } \{x_n + y_n\} \qquad \text{and} \qquad \text{(ii) } \{x_n y_n\}.$$

2. Prove that if $\{x_m\}$ and $\{y_m\}$ are Cauchy sequences in (S, ρ), then the sequence of distances

$$\rho(x_m, y_m), \quad m = 1, 2, \ldots,$$

converges in E^1.

[Hint: Show that this sequence is Cauchy in E^1; then use Theorem 4.]

3. Prove that a sequence $\{x_m\}$ is Cauchy in (S, ρ) iff

$$(\forall \varepsilon > 0)\ (\exists k)\ (\forall m > k) \quad \rho(x_m, x_k) < \varepsilon.$$

4. Two sequences $\{x_m\}$ and $\{y_m\}$ are called *concurrent* iff

$$\rho(x_m, y_m) \to 0.$$

Notation: $\{x_m\} \approx \{y_m\}$. Prove the following.

(i) If one of them is Cauchy or convergent, so is the other, and $\lim x_m = \lim y_m$ (if it exists).

[1] Here (S, ρ) *itself* need not be complete.

(ii) If any two sequences converge to the same limit, they are concurrent.

5. Show that if $\{x_m\}$ and $\{y_m\}$ are Cauchy sequences in (S, ρ), then

$$\lim_{m\to\infty} \rho(x_m, y_m)$$

does not change if $\{x_m\}$ or $\{y_m\}$ is replaced by a *concurrent* sequence (see Problems 4 and 2).

Call

$$\lim_{m\to\infty} \rho(x_m, y_m)$$

the "distance"

$$\rho(\{x_m\}, \{y_m\})$$

between $\{x_m\}$ and $\{y_m\}$. Prove that such "distances" satisfy all metric axioms, except that $\rho(\{x_m\}, \{y_m\})$ may be 0 even for *different* sequences. (When?)

Also, show that if

$$(\forall\, m) \quad x_m = a \text{ and } y_m = b \text{ (constant)},$$

then $\rho(\{x_m\}, \{y_m\}) = \rho(a, b)$.

5′. Continuing Problems 4 and 5, show that the concurrence relation ($\approx$) is reflexive, symmetric, and transitive (Chapter 1, §§4–7), i.e., an *equivalence relation*. That is, given $\{x_m\}$, $\{y_m\}$ in S, prove that

(a) $\{x_m\} \approx \{x_m\}$ (reflexivity);

(b) if $\{x_m\} \approx \{y_m\}$ then $\{y_m\} \approx \{x_m\}$ (symmetry);

(c) if $\{x_m\} \approx \{y_m\}$ and $\{y_m\} \approx \{z_m\}$, then $\{x_m\} \approx \{z_m\}$ (transitivity).

***5″.** From Problem 4 deduce that the set of all sequences in (S, ρ) splits into disjoint *equivalence* classes (as defined in Chapter 1, §§4–7) under the relation of concurrence ($\approx$). Show that all sequences of one and the same class either converge to the same limit or have no limit at all, and either none of them is Cauchy or all are Cauchy.

6. Give examples of incomplete metric spaces possessing complete subspaces.

7. Prove that if a sequence $\{x_m\} \subseteq (S, \rho)$ is Cauchy then it has a subsequence $\{x_{m_k}\}$ such that

$$(\forall\, k) \quad \rho(x_{m_k}, x_{m_{k+1}}) < 2^{-k}.$$

8. Show that every discrete space (S, ρ) is complete.

***9.** Let C be the set of all Cauchy sequences in (S, ρ); we denote them by capitals, e.g., $X = \{x_m\}$. Let

$$X^* = \{Y \in C \mid Y \approx X\}$$

denote the equivalence class of X under *concurrence*, $\approx$ (see Problems 2, $5'$, and $5''$). We define

$$\sigma(X^*, Y^*) = \rho(\{x_m\}, \{y_m\}) = \lim_{m\to\infty} \rho(x_m, y_m).$$

By Problem 5, this is *unambiguous*, for $\rho(\{x_m\}, \{y_m\})$ does not depend on the particular choice of $\{x_m\} \in X^*$ and $\{y_m\} \in Y^*$; and $\lim \rho(x_m, y_m)$ exists by Problem 2.

Show that σ *is a metric for the set of all equivalence classes* X^* $(X \in C)$; call this set C^*.

***10.** Continuing Problem 9, let x^* denote the equivalence class of the sequence *with all terms equal to* x; let C' be the set of all such "constant" equivalence classes (it is a subset of C^*).

Show that C' *is dense in* (C^*, σ), i.e., $\overline{C'} = C^*$ under the metric σ. (See §16, Definition 2.)

[Hint: Fix any "point" $X^* \in C^*$ and any globe $G(X^*; \varepsilon)$ about X^* in (C^*, σ). We must show that it contains some $x^* \in C'$.

By definition, X^* is the equivalence class of some Cauchy sequence $X = \{x_m\}$ in (S, ρ), so

$$(\exists\, k)\ (\forall\, m, n > k) \quad \rho(x_m, x_n) < \frac{\varepsilon}{2}.$$

Fix some $x = x_n$ $(n > k)$ and consider the equivalence class x^* of the sequence $\{x, x, \ldots, x, \ldots\}$; thus, $x^* \in C'$, and

$$\sigma(X^*, x^*) = \lim_{m\to\infty} \rho(x_m, x) \le \frac{\varepsilon}{2}. \quad \text{(Why?)}$$

Thus $x^* \in G(X^*, \varepsilon)$, as required.]

***11.** Two metric spaces (S, ρ) and (T, σ) are said to be *isometric* iff there is a map $f\colon S \underset{\text{onto}}{\longleftrightarrow} T$ such that

$$(\forall\, x, y \in S) \quad \rho(x, y) = \sigma(f(x), f(y)).$$

Show that *the spaces* (S, ρ) *and* (C', σ) *of Problem* 10 *are isometric*. Note that it is customary not to distinguish between two isometric spaces, treating each of them as just an "isometric copy" of the other. Indeed, distances in each of them are alike.

[Hint: Define $f(x) = x^*$.]

***12.** Continuing Problems 9 to 11, show that the space (C^*, σ) is complete. Thus prove that *for every metric space* (S, ρ), *there is a complete metric space* (C^*, σ) *containing an isometric copy* C' *of* S, *with* C' *dense in* C^*. C^* is called a *completion* of (S, ρ).

[Hint: Take a Cauchy sequence $\{X_m^*\}$ in (C^*, σ). By Problem 10, each globe $G(X_m^*; \frac{1}{m})$ contains some $x_m^* \in C'$, where x_m^* is the equivalence class of

$$\{x_m, x_m, \ldots, x_m, \ldots\},$$

and $\sigma(X_m^*, x_m^*) < \frac{1}{m} \to 0$. Thus by Problem 4, $\{x_m^*\}$ is Cauchy in (C^*, σ), as is $\{X_m^*\}$. Deduce that $X = \{x_m\} \in C$, and $X^* = \lim_{m\to\infty} X_m^*$ in (C^*, σ).]

Chapter 4

Function Limits and Continuity

§1. Basic Definitions

We shall now consider functions whose domains and ranges are sets in some fixed (but otherwise arbitrary) metric spaces (S, ρ) and (T, ρ'), respectively. We write

$$f\colon A \to (T, \rho')$$

for a function f with $D_f = A \subseteq (S, \rho)$ and $D'_f \subseteq (T, \rho')$. S is called the *domain space*, and T the *range space*, of f.

I. Given such a function, we often have to investigate its "*local behavior*" near some point $p \in S$. In particular, if $p \in A = D_f$ (so that $f(p)$ is *defined*) we may ask: Is it possible to make the function values $f(x)$ *as near as we like* ("ε-near") to $f(p)$ by keeping x *sufficiently close* ("δ-close") to p, i.e., inside some sufficiently small globe $G_p(\delta)$?[1] If this is the case, we say that f *is continuous at* p. More precisely, we formulate the following definition.

Definition 1.

A function $f\colon A \to (T, \rho')$, with $A \subseteq (S, \rho)$, is said to be *continuous at* p iff $p \in A$ and, moreover, *for each* $\varepsilon > 0$ (*no matter how small*) *there is* $\delta > 0$ *such that* $\rho'(f(x), f(p)) < \varepsilon$ *for all* $x \in A \cap G_p(\delta)$. In symbols,

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, x \in A \cap G_p(\delta)) \quad \begin{cases} \rho'(f(x), f(p)) < \varepsilon, \text{ or} \\ f(x) \in G_{f(p)}(\varepsilon). \end{cases} \tag{1}$$

If (1) *fails*, we say that f is *discontinuous* at p and call p a *discontinuity point* of f. This is also the case if $p \notin A$ (since $f(p)$ is *not defined*).

If (1) holds *for each* p *in a set* $B \subseteq A$, we say that f is continuous *on* B. If this is the case for $B = A$, we simply say that f *is continuous*.

[1] Of course, for $f(x)$ to exist, x must also be in $A = D_f$; thus $x \in A \cap G_p(\delta)$. We say that x is *δ-close to p* iff $\rho(x, p) < \delta$.

Sometimes we prefer to keep x near p but *different* from p. We then replace $G_p(\delta)$ in (1) by the set $G_p(\delta) - \{p\}$, i.e., the globe *without its center*, denoted $G_{\neg p}(\delta)$ and called the *deleted δ-globe* about p. This is even *necessary* if $p \notin D_f$. Replacing $f(p)$ in (1) by some $q \in T$, we then are led to the following definition.

Definition 2.

Given $f\colon A \to (T, \rho')$, $A \subseteq (S, \rho)$, $p \in S$, and $q \in T$, we say that $f(x)$ *tends to q as x tends to p* ($f(x) \to q$ as $x \to p$) iff *for each $\varepsilon > 0$ there is $\delta > 0$ such that $\rho'(f(x), q) < \varepsilon$ for all $x \in A \cap G_{\neg p}(\delta)$*. In symbols,

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, x \in A \cap G_{\neg p}(\delta)) \quad \begin{cases} \rho'(f(x), q) < \varepsilon, \text{ i.e.,} \\ f(x) \in G_q(\varepsilon). \end{cases} \tag{2}$$

This means that $f(x)$ is ε-close to q when x is δ-close to p *and* $x \neq p$.[2]

If (2) holds for some q, we call q a *limit of f at p*. There may be no such q. We then say that f has no limit at p, or that this limit does not exist. If there is *only one* such q (for a given p), we write $q = \lim_{x \to p} f(x)$.

Note 1. Formula (2) holds "vacuously" (see Chapter 1, §§1–3, end remark) if $A \cap G_{\neg p}(\delta) = \emptyset$ for some $\delta > 0$. Then *any* $q \in T$ is a limit at p, so a limit exists but is *not unique*. (We discard the case where T is a singleton.)

Note 2. However, *uniqueness is ensured if $A \cap G_{\neg p}(\delta) \neq \emptyset$ for all $\delta > 0$*, as we prove below.

Observe that by Corollary 6 of Chapter 3, §14, *the set A clusters at p iff*

$$(\forall\, \delta > 0) \quad A \cap G_{\neg p}(\delta) \neq \emptyset. \quad \text{(Explain!)}$$

Thus we have the following corollary.

Corollary 1. *If A clusters at p in (S, ρ), then a function $f\colon A \to (T, p')$ can have at most one limit at p; i.e.,*

$$\lim_{x \to p} f(x) \textit{ is unique (if it exists).}^{3}$$

In particular, this holds if $A \supseteq (a, b) \subset E^1$ $(a < b)$ and $p \in [a, b]$.

Proof. Suppose f has *two* limits, q and r, at p. By the Hausdorff property,

$$G_q(\varepsilon) \cap G_r(\varepsilon) = \emptyset \quad \text{for some } \varepsilon > 0.$$

Also, by (2), there are $\delta', \delta'' > 0$ such that

$$\begin{aligned} (\forall\, x \in A \cap G_{\neg p}(\delta')) \quad & f(x) \in G_q(\varepsilon) \text{ and} \\ (\forall\, x \in A \cap G_{\neg p}(\delta'')) \quad & f(x) \in G_r(\varepsilon). \end{aligned}$$

[2] Observe that the choice of δ *depends on* ε in both (1) and (2).

[3] Because of this, some authors *restrict* Definition 2 to the case where A clusters at p. However, this has its disadvantages (e.g., Corollary 2 fails).

Let $\delta = \min(\delta', \delta'')$. Then for $x \in A \cap G_{\neg p}(\delta)$, $f(x)$ is in *both* $G_q(\varepsilon)$ and $G_r(\varepsilon)$, and such an x *exists* since $A \cap G_{\neg p}(\delta) \neq \emptyset$ by assumption.

But this is impossible since $G_q(\varepsilon) \cap G_r(\varepsilon) = \emptyset$ (a contradiction!). □

For *intervals*, see Chapter 3, §14, Example (h).

Corollary 2. *f is continuous at p ($p \in D_f$) iff $f(x) \to f(p)$ as $x \to p$.*

The straightforward proof from definitions is left to the reader.

Note 3. In formula (2), we *excluded the case* $x = p$ by assuming that $x \in A \cap G_{\neg p}(\delta)$. This makes the behavior of f *at p itself* irrelevant. Thus *for the existence of a limit q at p, it does not matter whether $p \in D_f$ or whether* $f(p) = q$. But *both conditions are required for continuity at p* (see Corollary 2 and Definition 1).

Note 4. Observe that if (1) or (2) holds for some δ, it certainly holds for any $\delta' \leq \delta$. *Thus we may always choose δ as small as we like.* Moreover, as x is limited to $G_p(\delta)$, we may disregard, or change at will, the function values $f(x)$ for $x \notin G_p(\delta)$ ("*local character of the limit notion*").

II. Limits in E^*. If S or T is E^* (or E^1), we may let $x \to \pm\infty$ or $f(x) \to \pm\infty$. For a precise definition, we rewrite (2) in terms of *globes* G_p and G_q:

$$(\forall\, G_q)\ (\exists\, G_p)\ (\forall\, x \in A \cap G_{\neg p}) \quad f(x) \in G_q. \tag{2'}$$

This makes sense also if $p = \pm\infty$ or $q = \pm\infty$. We only have to use our conventions as to $G_{\pm\infty}$, or the metric ρ' for E^*, as explained in Chapter 3, §11.

For example, consider

$$\text{“}f(x) \to q \text{ as } x \to +\infty\text{”}\ (A \subseteq S = E^*,\, p = +\infty,\, q \in (T,\, \rho')).$$

Here G_p has the form $(a, +\infty]$, $a \in E^1$, and $G_{\neg p} = (a, +\infty)$, while $G_q = G_q(\varepsilon)$, as usual. Noting that $x \in G_{\neg p}$ means $x > a$ ($x \in E^1$), we can rewrite (2′) as

$$(\forall\, \varepsilon > 0)\ (\exists\, a \in E^1)\ (\forall\, x \in A \mid x > a) \quad f(x) \in G_q(\varepsilon), \text{ or } \rho'(f(x),\, q) < \varepsilon. \tag{3}$$

This means that $f(x)$ becomes arbitrarily close to q *for large x* ($x > a$).

Next consider "$f(x) \to +\infty$ as $x \to -\infty$." Here $G_{\neg p} = (-\infty, a)$ and $G_q = (b, +\infty]$. Thus formula (2′) yields (with $S = T = E^*$, and x varying over E^1)

$$(\forall\, b \in E^1)\ (\exists\, a \in E^1)\ (\forall\, x \in A \mid x < a) \quad f(x) > b; \tag{4}$$

similarly in other cases, which we leave to the reader.

Note 5. In (3), we may take $A = N$ (the naturals). Then $f\colon N \to (T, \rho')$ is a *sequence* in T. Writing m for x, set $u_m = f(m)$ and $a = k \in N$ to obtain

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, m > k) \quad u_m \in G_q(\varepsilon);\ \text{i.e., } \rho'(u_m,\, q) < \varepsilon.$$

This coincides with our definition of the limit q of a sequence $\{u_m\}$ (see Chapter 3, §14). *Thus limits of sequences are a special case of function limits.* Theorems on sequences can be obtained from those on functions $f\colon A \to (T, \rho')$ by simply taking $A = N$ and $S = E^*$ as above.

Note 6. Formulas (3) and (4) make sense also if $S = E^1$ (respectively, $S = T = E^1$) since they *do not* involve any mention of $\pm\infty$. We shall use such formulas also for functions $f\colon A \to T$, with $A \subseteq S \subseteq E^1$ or $T \subseteq E^1$, as the case may be.

III. Relative Limits and Continuity. Sometimes the desired result (1) or (2) does not hold *in full*, but only *with A replaced by a smaller set $B \subseteq A$.* Thus we may have

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, x \in B \cap G_{\neg p}(\delta)) \quad f(x) \in G_q(\varepsilon).$$

In this case, we call q a *relative limit* of f at p *over* B and write

$$\text{“}f(x) \to q \text{ as } x \to p \textit{ over } B\text{”}$$

or

$$\lim_{x \to p,\, x \in B} f(x) = q \quad \text{(if } q \text{ is unique);}$$

B is called the *path* over which x tends to p. If, in addition, $p \in D_f$ and $q = f(p)$, we say that f is *relatively continuous at* p over B; then (1) holds with A replaced by B. Again, if this holds for *every* $p \in B$, we say that f is *relatively continuous on* B. Clearly, if $B = A = D_f$, this yields *ordinary* (nonrelative) limits and continuity. Thus relative limits and continuity are *more general*.

Note that for limits over a path B, x is chosen from B or $B - \{p\}$ only. Thus the behavior of f outside B becomes irrelevant, and so we may arbitrarily redefine f on $-B$. For example, if $p \notin B$ but $\lim_{x\to p,\, x\in B} f(x) = q$ exists, we may define $f(p) = q$, thus making f relatively continuous at p (over B). We also may replace (S, ρ) by (B, ρ) (if $p \in B$), or *restrict f to B*, i.e., replace f by the function $g\colon B \to (T, \rho')$ defined by $g(x) = f(x)$ for $x \in B$ (briefly, $g = f$ *on* B).[4]

A particularly important case is

$$A \subseteq S \subseteq E^*, \text{ e.g., } S = E^1.$$

Then *inequalities* are defined in S, so we may take

$$B = \{x \in A \mid x < p\} \text{ (points in } A\text{, preceding } p\text{).}$$

[4] The function g is called the restriction of f to B denoted f_B or $f|_B$. Thus f is *relatively* continuous on B iff f_B is continuous.

Then, writing G_q for $G_q(\varepsilon)$ and $a = p - \delta$, we obtain from formula (2)

$$(\forall\, G_q)\ (\exists\, a < p)\ (\forall\, x \in A \mid a < x < p)\quad f(x) \in G_q. \tag{5}$$

If (5) holds, we call q a *left* limit of f at p and write

"$f(x) \to q$ as $x \to p^-$" ("x tends to p *from the left*").

If, in addition, $q = f(p)$, we say that f is *left continuous* at p. Similarly, taking

$$B = \{x \in A \mid x > p\},$$

we obtain *right* limits and continuity. We write

$$f(x) \to q \ \text{as}\ x \to p^+$$

iff q is a *right* limit of f at p, i.e., if (5) holds with all inequalities reversed.

If the set B in question *clusters at* p, the relative limit (if any) is *unique*. We then denote the left and right limit, respectively, by $f(p^-)$ and $f(p^+)$, and we write

$$\lim_{x \to p^-} f(x) = f(p^-) \text{ and } \lim_{x \to p^+} f(x) = f(p^+). \tag{6}$$

Corollary 3. *With the previous notation, if $f(x) \to q$ as $x \to p$ over a path B, and also over D, then $f(x) \to q$ as $x \to p$ over $B \cup D$.*

Hence if $D_f \subseteq E^*$ and $p \in E^*$, we have

$$q = \lim_{x \to p} f(x) \ \textit{iff}\ q = f(p^-) = f(p^+). \quad \text{(Exercise!)}$$

We now illustrate our definitions by a diagram in E^2 representing a function $f\colon E^1 \to E^1$ by its *graph*, i.e., points (x, y) such that $y = f(x)$.

Here

$$G_q(\varepsilon) = (q - \varepsilon,\, q + \varepsilon)$$

is an interval on the *y-axis*. The dotted lines show how to construct an interval

$$(p - \delta,\, p + \delta) = G_p$$

on the *x-axis*, satisfying formula (1) in Figure 13, formulas (5) and (6) in Figure 14, or formula (2) in Figure 15. The point Q in each diagram *belongs to the graph*; i.e., $Q = (p, f(p))$. In Figure 13, f is continuous at p (and also at p_1). However, it is only *left*-continuous at p in Figure 14, and it is discontinuous at p in Figure 15, though $f(p^-)$ and $f(p^+)$ exist. (Why?)

Examples.

(a) Let $f\colon A \to T$ be *constant* on $B \subseteq A$; i.e.,

$$f(x) = q \text{ for a fixed } q \in T \text{ and all } x \in B.$$

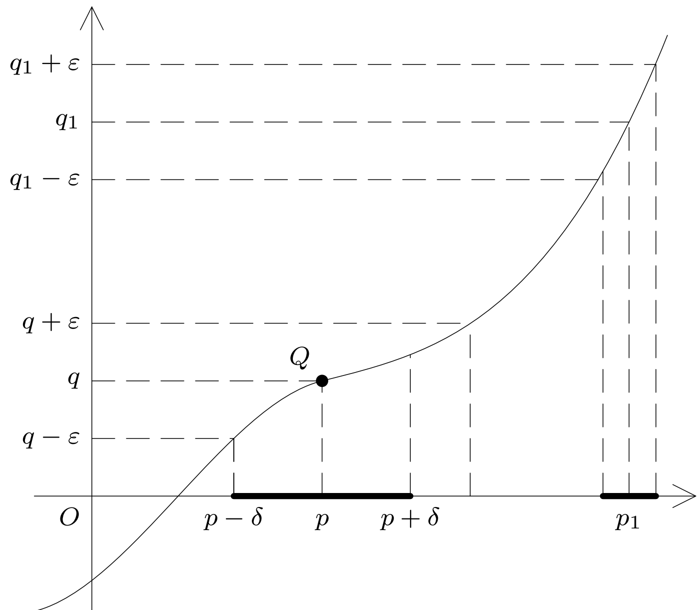

FIGURE 13

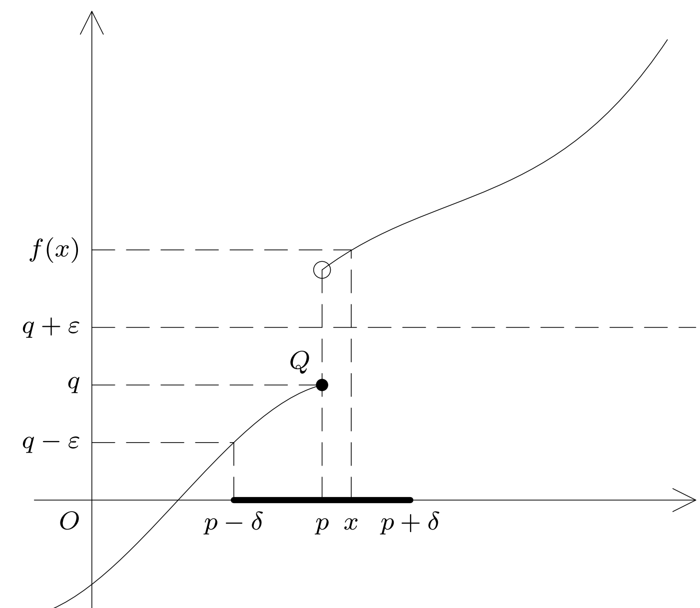

FIGURE 14

Then f *is relatively continuous on* B, *and* $f(x) \to q$ *as* $x \to p$ *over* B, at each p. (Given $\varepsilon > 0$, take an arbitrary $\delta > 0$. Then

$$(\forall\, x \in B \cap G_{\neg p}(\delta)) \quad f(x) = q \in G_q(\varepsilon),$$

as required; similarly for continuity.)

(b) Let f be the *identity map* on $A \subset (S,\, \rho)$; i.e.,

$$(\forall\, x \in A) \quad f(x) = x.$$

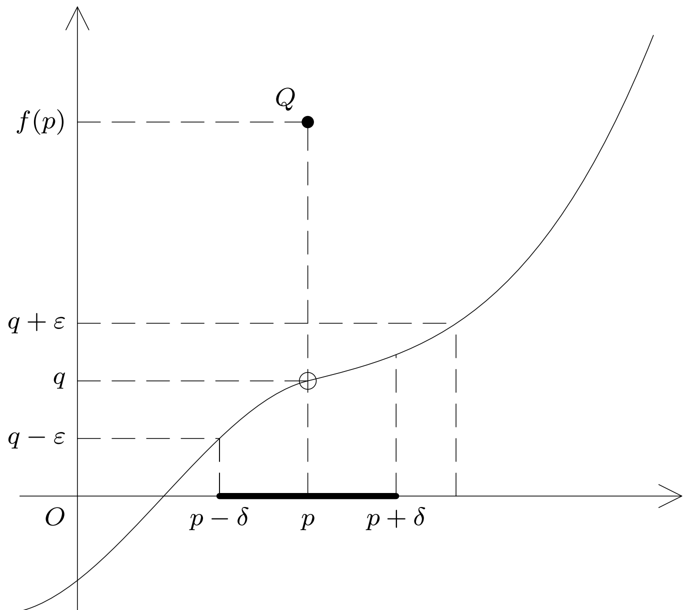

FIGURE 15

Then, given $\varepsilon > 0$, take $\delta = \varepsilon$ to obtain, for $p \in A$,

$$(\forall\, x \in A \cap G_p(\delta)) \quad \rho(f(x), f(p)) = \rho(x, p) < \delta = \varepsilon.$$

Thus by (1), f is continuous at any $p \in A$, hence *on* A.

(c) Define $f\colon E^1 \to E^1$ by

$$f(x) = 1 \text{ if } x \text{ is rational, and } f(x) = 0 \text{ otherwise.}$$

(This is the *Dirichlet function*, so named after Johann Peter Gustav Lejeune Dirichlet.)

No matter how small δ is, the globe

$$G_p(\delta) = (p - \delta, p + \delta)$$

(even the *deleted* globe) contains both rationals and irrationals. Thus as x varies over $G_{\neg p}(\delta)$, $f(x)$ takes on *both* values, 0 and 1, many times and so *gets out of any* $G_q(\varepsilon)$, with $q \in E^1$, $\varepsilon < \frac{1}{2}$.

Hence for any $q, p \in E^1$, formula (2) *fails* if we take $\varepsilon = \frac{1}{4}$, say. Thus f has no limit at any $p \in E^1$ and hence is discontinuous everywhere! However, f is *relatively* continuous on the set R of all rationals by Example (a).

(d) Define $f\colon E^1 \to E^1$ by

$$f(x) = [x] \text{ (= the integral part of } x\text{; see Chapter 2, §10).}$$

Thus $f(x) = 0$ for $x \in [0, 1)$, $f(x) = 1$ for $x \in [1, 2)$, etc. Then f is discontinuous at p if p is an *integer* (why?) but continuous at any other p (restrict f to a small $G_p(\delta)$ so as to make it constant).

However, left and right limits exist at *each* $p \in E^1$, even if $p = n$ (an integer). In fact,

$$f(x) = n,\ x \in (n,\, n+1)$$

and

$$f(x) = n - 1,\ x \in (n-1,\, n),$$

hence $f(n^+) = n$ and $f(n^-) = n - 1$; f is *right continuous on* E^1. See Figure 16.

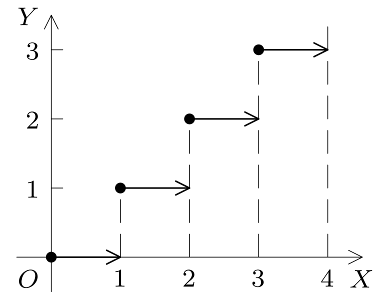

FIGURE 16

(e) Define $f\colon E^1 \to E^1$ by

$$f(x) = \frac{x}{|x|} \text{ if } x \neq 0, \text{ and } f(0) = 0.$$

(This is the so-called *signum function*, often denoted by sgn.)

Then (Figure 17)

$$f(x) = -1 \text{ if } x < 0$$

and

$$f(x) = 1 \text{ if } x > 0.$$

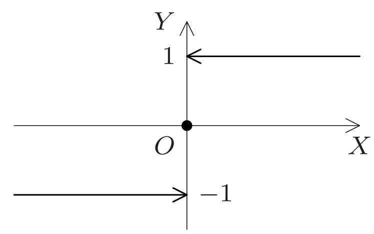

FIGURE 17

Thus, as in (d), we infer that f is discontinuous at 0, but continuous at each $p \neq 0$. Also, $f(0^+) = 1$ and $f(0^-) = -1$. Redefining $f(0) = 1$ or $f(0) = -1$, we can make f right (respectively, left) continuous at 0, but not both.

(f) Define $f\colon E^1 \to E^1$ by (see Figure 18)

$$f(x) = \sin\frac{1}{x} \text{ if } x \neq 0, \text{ and } f(0) = 0.$$

Any globe $G_0(\delta)$ about 0 contains points at which $f(x) = 1$, as well as those at which $f(x) = -1$ or $f(x) = 0$ (take $x = 2/(n\pi)$ for large integers n); in fact, the graph "oscillates" infinitely many times between -1 and 1. Thus by the same argument as in (c), f has no limit at

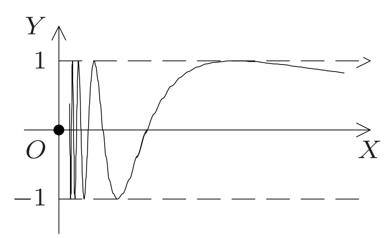

FIGURE 18

0 (not even a left or right limit) and hence is discontinuous at 0. No attempt at redefining f at 0 can restore even left or right continuity, let alone ordinary continuity, at 0.

(g) Define $f\colon E^2 \to E^1$ by

$$f(\bar{0}) = 0 \text{ and } f(\bar{x}) = \frac{x_1 x_2}{x_1^2 + x_2^2} \text{ if } \bar{x} = (x_1, x_2) \neq \bar{0}.$$

Let B be any *line* in E^2 through $\bar{0}$, given parametrically by

$$\bar{x} = t\vec{u}, \quad t \in E^1, \ \vec{u} \text{ fixed (see Chapter 3, §§4–6)},$$

so $x_1 = tu_1$ and $x_2 = tu_2$. As is easily seen, for $\bar{x} \in B$, $f(\bar{x}) = f(\bar{u})$ (*constant*) if $\bar{x} \neq \bar{0}$. Hence

$$(\forall \bar{x} \in B \cap G_{\neg\bar{0}}(\delta)) \quad f(\bar{x}) = f(\bar{u}),$$

i.e., $\rho(f(\bar{x}), f(\bar{u})) = 0 < \varepsilon$, for any $\varepsilon > 0$ and any deleted globe about $\bar{0}$.

By (2′), then, $f(\bar{x}) \to f(\bar{u})$ as $\bar{x} \to \bar{0}$ over the path B. Thus f has a *relative* limit $f(\bar{u})$ at $\bar{0}$, over any line $\bar{x} = t\bar{u}$, but *this limit is different for various choices of* $\bar{u}$, i.e., for different lines through $\bar{0}$. No *ordinary* limit at $\bar{0}$ exists (why?); f is not even *relatively* continuous at $\bar{0}$ over the line $\bar{x} = t\vec{u}$ unless $f(\bar{u}) = 0$ (which is the case only if the line is one of the coordinate axes (why?)).

Problems on Limits and Continuity

1. Prove Corollary 2. Why can one interchange $G_p(\delta)$ and $G_{\neg p}(\delta)$ here?

2. Prove Corollary 3. By induction, extend its first clause to unions of n paths. Disprove it for *infinite* unions of paths (see Problem 9 in §3).

2′. Prove that a function $f\colon E^1 \to (T, \rho')$ is continuous at p iff

$$f(p) = f(p^-) = f(p^+).$$

3. Show that relative limits and continuity at p (over B) are equivalent to the *ordinary* ones if B is a *neighborhood* of p (Chapter 3, §12); for example, if it is some G_p.

4. Discuss Figures 13–15 in detail, comparing $f(p)$, $f(p^-)$, and $f(p^+)$; see Problem 2′.

Observe that in Figure 13, *different* values of δ result at p and p_1 for the *same* ε. Thus δ depends on *both* ε and the choice of p.

5. Complete the missing details in Examples (d)–(g). In (d), redefine $f(x)$ to be the *least integer* $\geq x$. Show that f is then *left*-continuous on E^1.

6. Give explicit definitions (such as (3)) for

(a) $\lim_{x\to+\infty} f(x) = -\infty$; (b) $\lim_{x\to-\infty} f(x) = q$;

(c) $\lim_{x\to p} f(x) = +\infty$; (d) $\lim_{x\to p} f(x) = -\infty$;

(e) $\lim_{x\to p^-} f(x) = +\infty$; (f) $\lim_{x\to p^+} f(x) = -\infty$.

In each case, draw a diagram (such as Figures 13–15) and determine whether the domain and range of f *must both* be in E^*.

7. Define $f\colon E^1 \to E^1$ by

$$f(x) = \frac{x^2-1}{x-1} \text{ if } x \neq 1, \text{ and } f(1) = 0.$$

Show that $\lim_{x\to 1} f(x) = 2$ exists, yet f is discontinuous at $p = 1$. Make it continuous by redefining $f(1)$.

[Hint: For $x \neq 1$, $f(x) = x + 1$. Proceed as in Example (b), using the *deleted* globe $G_{\neg p}(\delta)$.]

8. Find $\lim_{x\to p} f(x)$ and check continuity at p in the following cases, assuming that $D_f = A$ is the set of *all* $x \in E^1$ for which the given expression for $f(x)$ has sense. Specify that set.[5]

(a) $\lim_{x\to 2}(2x^2 - 3x - 5)$; (b) $\lim_{x\to 1} \frac{3x+2}{2x-1}$;

(c) $\lim_{x\to -1}\left(\frac{x^2-4}{x+2} - 1\right)$; (d) $\lim_{x\to 2} \frac{x^3-8}{x-2}$;

(e) $\lim_{x\to a} \frac{x^4-a^4}{x-a}$; (f) $\lim_{x\to 0}\left(\frac{x}{x+1}\right)^3$;

(g) $\lim_{x\to -1}\left(\frac{1}{x^2+1}\right)^2$.

[Example solution: Find $\lim_{x\to 1} \frac{5x^2-1}{2x+3}$.

Here

$$f(x) = \frac{5x^2-1}{2x+3};\ A = E^1 - \left\{-\frac{3}{2}\right\};\ p = 1.$$

We show that f is *continuous* at p, and so (by Corollary 2)

$$\lim_{x\to p} f(x) = f(p) = f(1) = \frac{4}{5}.$$

Using formula (1), we fix an arbitrary $\varepsilon > 0$ and look for a δ such that

$$(\forall x \in A \cap G_p(\delta)) \quad \rho(f(x), f(1)) = |f(x) - f(1)| < \varepsilon, \text{ i.e., } \left|\frac{5x^2-1}{2x+3} - \frac{4}{5}\right| < \varepsilon;$$

[5] In (d) and (e), $p \notin A$, yet one can restore continuity as in Problem 7. (Reduce the fraction by $x - p$ for $x \neq p$ and define $f(p)$ accordingly.)

or, by putting everything over a common denominator and using properties of absolute values,

$$|x-1|\,\frac{|25x+17|}{5|2x+3|}<\varepsilon \text{ whenever } |x-1|<\delta \text{ and } x\in A. \tag{6}$$

(Usually in such problems, it is desirable to *factor out* $x-p$.)

By Note 4, we may assume $0<\delta\le 1$. Then $|x-1|<\delta$ implies $-1\le x-1\le 1$, i.e., $0\le x\le 2$, so

$$5\,|2x+3|\ge 15 \text{ and } |25x+17|\le 67.$$

Hence (6) will certainly hold if

$$|x-1|\,\frac{67}{15}<\varepsilon, \text{ i.e., if } |x-1|<\frac{15\varepsilon}{67}.$$

To achieve it, we choose $\delta=\min(1,\,15\varepsilon/67)$. Then, reversing all steps, we obtain (6), and hence $\lim_{x\to 1} f(x)=f(1)=4/5$.]

9. Find (using definitions, such as (3))

(a) $\displaystyle\lim_{x\to+\infty}\frac{1}{x}$; (b) $\displaystyle\lim_{x\to-\infty}\frac{3x+2}{2x-1}$;

(c) $\displaystyle\lim_{x\to+\infty}\frac{x^3}{1-x^2}$; (d) $\displaystyle\lim_{x\to 3^+}\frac{x-1}{x-3}$;

(e) $\displaystyle\lim_{x\to 3^-}\frac{x-1}{x-3}$; (f) $\displaystyle\lim_{x\to 3}\left|\frac{x-1}{x-3}\right|$.

10. Prove that if

$$\lim_{x\to p} f(x)=\bar{q}\in E^n\ ({}^*C^n),$$

then for each scalar c,

$$\lim_{x\to p} cf(x)=c\bar{q}.$$

11. Define $f\colon E^1\to E^1$ by

$$f(x)=x\cdot\sin\frac{1}{x} \text{ if } x\neq 0, \text{ and } f(0)=0.$$

Show that f is continuous at $p=0$, i.e.,

$$\lim_{x\to 0} f(x)=f(0)=0.$$

Draw an approximate graph (it is contained between the lines $y=\pm x$).

[Hint: $\left|x\cdot\sin\frac{1}{x}-0\right|\le|x|$.]

***12.** Discuss the statement: f is continuous at p iff

$$(\forall\, G_{f(p)})\ (\exists\, G_p)\quad f[G_p]\subseteq G_{f(p)}.$$

13. Define $f\colon E^1 \to E^1$ by

$$f(x) = x \text{ if } x \text{ is rational}$$

and

$$f(x) = 0 \text{ otherwise.}$$

Show that f is continuous at 0 but nowhere else. How about *relative* continuity?

14. Let $A = (0, +\infty) \subset E^1$. Define $f\colon A \to E^1$ by

$$f(x) = 0 \text{ if } x \text{ is irrational}$$

and

$$f(x) = \frac{1}{n} \text{ if } x = \frac{m}{n} \ (\textit{in lowest terms})$$

for some natural m and n. Show that f is continuous at each irrational, but at no rational, point $p \in A$.

[Hints: If p *is irrational*, fix $\varepsilon > 0$ and an integer $k > 1/\varepsilon$. In $G_p(1)$, there are only finitely many irreducible fractions

$$\frac{m}{n} > 0 \text{ with } n \leq k,$$

so one of them, call it r, is *closest* to p. Put

$$\delta = \min(1, |r - p|)$$

and show that

$$(\forall x \in A \cap G_p(\delta)) \quad |f(x) - f(p)| = f(x) < \varepsilon,$$

distinguishing the cases where x is rational and irrational.

If p *is rational*, use the fact that each $G_p(\delta)$ contains irrationals x at which

$$f(x) = 0 \Longrightarrow |f(x) - f(p)| = f(p).$$

Take $\varepsilon < f(p)$.]

15. Given two reals, $p > 0$ and $q > 0$, define $f\colon E^1 \to E^1$ by

$$f(0) = 0 \text{ and } f(x) = \left(\frac{x}{p}\right) \cdot \left[\frac{q}{x}\right] \text{ if } x \neq 0;$$

here $[q/x]$ is the integral part of q/x.

(i) Is f left or right continuous at 0?

(ii) Same question with $f(x) = [x/p](q/x)$.

16. Prove that if (S, ρ) is discrete, then *all* functions $f\colon S \to (T, \rho')$ are continuous. What if (T, ρ') is discrete but (S, ρ) is not?

§2. Some General Theorems on Limits and Continuity

I. In §1 we gave the so-called "ε, δ" definition of continuity. Now we present another (equivalent) formulation, known as the *sequential* one. Roughly, it states that f is continuous iff it *carries convergent sequences* $\{x_m\} \subseteq D_f$ *into convergent "image sequences"* $\{f(x_m)\}$. More precisely, we have the following theorem.

Theorem 1 (sequential criterion of continuity)**.** (i) *A function*

$$f\colon A \to (T,\, \rho'),\ \textit{with}\ A \subseteq (S,\, \rho),$$

is continuous at a point $p \in A$ *iff for every sequence* $\{x_m\} \subseteq A$ *such that* $x_m \to p$ *in* $(S,\, \rho)$*, we have* $f(x_m) \to f(p)$ *in* $(T,\, \rho')$*. In symbols,*

$$(\forall\, \{x_m\} \subseteq A \mid x_m \to p) \quad f(x_m) \to f(p). \tag{1$'$}$$

(ii) *Similarly, a point* $q \in T$ *is a limit of* f *at* p $(p \in S)$ *iff*

$$(\forall\, \{x_m\} \subseteq A - \{p\} \mid x_m \to p) \quad f(x_m) \to q. \tag{2$'$}$$

Note that in (2′) we consider only sequences of terms *other than* p.

Proof. We first prove (ii). Suppose q is a limit of f at p, i.e. (see §1),

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, x \in A \cap G_{\neg p}(\delta)) \quad f(x) \in G_q(\varepsilon). \tag{2}$$

Thus, given $\varepsilon > 0$, there is $\delta > 0$ (henceforth fixed) such that

$$f(x) \in G_q(\varepsilon) \text{ whenever } x \in A,\ x \neq p, \text{ and } x \in G_p(\delta). \tag{3}$$

We want to deduce (2′). Thus we fix any sequence

$$\{x_m\} \subseteq A - \{p\},\ x_m \to p.^1$$

Then

$$(\forall\, m) \quad x_m \in A \text{ and } x_m \neq p,$$

and $G_p(\delta)$ contains *all but finitely many* x_m. Then these x_m satisfy the conditions stated in (3). Hence $f(x_m) \in G_q(\varepsilon)$ *for all but finitely many* m. As ε is arbitrary, this implies $f(x_m) \to q$ (by the definition of $\lim_{m\to\infty} f(x_m)$), as is required in (2′). Thus (2) $\Longrightarrow$ (2′).

Conversely, suppose (2) *fails*, i.e., its negation holds. (See the rules for forming *negations* of such formulas in Chapter 1, §§1–3.) Thus

$$(\exists\, \varepsilon > 0)\ (\forall\, \delta > 0)\ (\exists\, x \in A \cap G_{\neg p}(\delta)) \quad f(x) \notin G_q(\varepsilon) \tag{4}$$

[1] If no such sequence exists, then (2′) is *vacuously* true and there is nothing to prove.

by the rules for quantifiers. We fix an ε satisfying (4), and let

$$\delta_m = \frac{1}{m}, \quad m = 1, 2, \ldots.$$

By (4), for each δ_m there is x_m (*depending on* δ_m) such that

$$x_m \in A \cap G_{\neg p}\Big(\frac{1}{m}\Big) \tag{5}$$

and

$$f(x_m) \notin G_q(\varepsilon), \quad m = 1, 2, 3, \ldots \tag{6}$$

We fix these x_m. As $x_m \in A$ and $x_m \neq p$, we obtain a sequence

$$\{x_m\} \subseteq A - \{p\}.$$

Also, as $x_m \in G_p(\frac{1}{m})$, we have $\rho(x_m, p) < 1/m \to 0$, and hence $x_m \to p$. On the other hand, by (6), the image sequence $\{f(x_m)\}$ *cannot* converge to q (why?), i.e., ($2'$) *fails*. Thus we see that ($2'$) *fails or holds accordingly as* (2) *does*.

This proves assertion (ii). Now, by setting $q = f(p)$ in (2) and ($2'$), we also obtain the *first* clause of the theorem, as to continuity. $\square$

Note 1. The theorem also applies to *relative* limits and continuity over a path B (just replace A by B in the proof), as well as to the cases $p = \pm\infty$ and $q = \pm\infty$ in E^* (for E^* can be treated as a *metric space*; see the end of Chapter 3, §11).

If the range space (T, ρ') is *complete* (Chapter 3, §17), then the image sequences $\{f(x_m)\}$ converge iff they are *Cauchy*. This leads to the following corollary.

Corollary 1. *Let* (T, ρ') *be complete, such as* E^n. *Let a map* $f\colon A \to T$ *with* $A \subseteq (S, \rho)$ *and a point* $p \in S$ *be given. Then for* f *to have a limit at* p, *it suffices that* $\{f(x_m)\}$ *be Cauchy in* (T, ρ') *whenever* $\{x_m\} \subseteq A - \{p\}$ *and* $x_m \to p$ *in* (S, ρ).

Indeed, as noted above, all such $\{f(x_m)\}$ *converge*. Thus it only remains to show that they tend to *one and the same* limit q, as is required in part (ii) of Theorem 1. We leave this as an exercise (Problem 1 below).

***Theorem 2** (Cauchy criterion for *functions*). *With the assumptions of Corollary* 1, *the function* f *has a limit at* p *iff for each* $\varepsilon > 0$, *there is* $\delta > 0$ *such that*

$$\rho'(f(x), f(x')) < \varepsilon \text{ for all } x, x' \in A \cap G_{\neg p}(\delta).$$[2]

In symbols,

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, x, x' \in A \cap G_{\neg p}(\delta)) \quad \rho'(f(x), f(x')) < \varepsilon. \tag{7}$$

[2] That is, $f(x)$ is ε-close to $f(x')$ when x and x' are δ-close to p, *but not equal to* p.

Proof. Assume (7). To show that f has a limit at p, we use Corollary 1. Thus we take any sequence

$$\{x_m\} \subseteq A - \{p\} \text{ with } x_m \to p$$

and show that $\{f(x_m)\}$ is Cauchy, i.e.,

$$(\forall \varepsilon > 0)\ (\exists k)\ (\forall m, n > k) \quad \rho'(f(x_m), f(x_n)) < \varepsilon.$$

To do this, fix an arbitrary $\varepsilon > 0$. By (7), we have

$$(\forall x, x' \in A \cap G_{\neg p}(\delta)) \quad \rho'(f(x), f(x')) < \varepsilon, \tag{7'}$$

for some $\delta > 0$. Now as $x_m \to p$, there is k such that

$$(\forall m, n > k) \quad x_m, x_n \in G_p(\delta).$$

As $\{x_m\} \subseteq A - \{p\}$, we even have $x_m, x_n \in A \cap G_{\neg p}(\delta)$. Hence by (7′),

$$(\forall m, n > k) \quad \rho'(f(x_m), f(x_n)) < \varepsilon;$$

i.e., $\{f(x_m)\}$ *is Cauchy*, as required in Corollary 1, and so f has a limit at p. This shows that (7) implies the existence of that limit.

The easy converse proof is left to the reader. (See Problem 2.) □

II. Composite Functions. The *composite* of two functions

$$f\colon S \to T \text{ and } g\colon T \to U,$$

denoted

$$g \circ f \quad \text{(in that order)},$$

is by definition *a map of S into U* given by

$$(g \circ f)(x) = g(f(x)), \quad s \in S.$$

Our next theorem states, roughly, that $g \circ f$ *is continuous if g and f are*. We shall use Theorem 1 to prove it.

Theorem 3. *Let (S, ρ), (T, ρ'), and (U, ρ'') be metric spaces. If a function $f\colon S \to T$ is continuous at a point $p \in S$, and if $g\colon T \to U$ is continuous at the point $q = f(p)$, then the composite function $g \circ f$ is continuous at p.*

Proof. The domain of $g \circ f$ is S. So take any sequence

$$\{x_m\} \subseteq S \text{ with } x_m \to p.$$

As f is continuous at p, formula (1′) yields $f(x_m) \to f(p)$, where $f(x_m)$ is in $T = D_g$. Hence, as g is continuous at $f(p)$, we have

$$g(f(x_m)) \to g(f(p)), \text{ i.e., } (g \circ f)(x_m) \to (g \circ f)(p),$$

and this holds for *any* $\{x_m\} \subseteq S$ with $x_m \to p$. Thus $g \circ f$ satisfies condition (1′) and is continuous at p. □

Caution: The fact that

$$\lim_{x\to p} f(x) = q \text{ and } \lim_{y\to q} g(y) = r$$

does not imply

$$\lim_{x\to p} g(f(x)) = r$$

(see Problem 3 for counterexamples).

Indeed, if $\{x_m\} \subseteq S-\{p\}$ and $x_m \to p$, we obtain, as before, $f(x_m) \to q$, but *not* $f(x_m) \neq q$. Thus we cannot *re-apply* formula $(2')$ to obtain $g(f(x_m)) \to r$ since $(2')$ *requires that* $f(x_m) \neq q$. The argument still works if g is continuous at q (then $(1')$ applies) or if $f(x)$ *never equals* q (then $f(x_m) \neq q$). It even suffices that $f(x) \neq q$ *for* x *in some deleted globe about* p (see §1, Note 4). Hence we obtain the following corollary.

Corollary 2. *With the notation of Theorem* 3, *suppose*

$$f(x) \to q \text{ as } x \to p, \text{ and } g(y) \to r \text{ as } y \to q.$$

Then

$$g(f(x)) \to r \text{ as } x \to p,$$

provided, however, that

(i) *g is continuous at q, or*

(ii) *$f(x) \neq q$ for x in some deleted globe about p, or*

(iii) *f is one to one, at least when restricted to some $G_{\neg p}(\delta)$.*

Indeed, (i) and (ii) suffice, as was explained above. Thus assume (iii). Then f can take the value q *at most once*, say, at some point

$$x_0 \in G_{\neg p}(\delta).$$

As $x_0 \neq p$, let

$$\delta' = \rho(x_0, p) > 0.$$

Then $x_0 \notin G_{\neg p}(\delta')$, *so* $f(x) \neq q$ *on* $G_{\neg p}(\delta')$, and case (iii) reduces to (ii).

We now show how to *apply* Corollary 2.

Note 2. Suppose we *know* that

$$r = \lim_{y\to q} g(y) \text{ exists.}$$

Using this fact, we often pass to another variable x, setting $y = f(x)$ where f is such that $q = \lim_{x\to p} f(x)$ for some p. We shall say that the substitution (or "change of variable") $y = f(x)$ is *admissible* if one of the conditions (i), (ii), or (iii) of Corollary 2 holds.[3] Then by Corollary 2,

$$\lim_{y \to q} g(y) = r = \lim_{x \to p} g(f(x))$$

(yielding *the second* limit).

Examples.

(A) Let

$$h(x) = \left(1 + \frac{1}{x}\right)^x \text{ for } |x| \geq 1.$$

Then

$$\lim_{x \to +\infty} h(x) = e.$$

For a proof, let $n = f(x) = [x]$ be the integral part of x. Then for $x > 1$,

$$\left(1 + \frac{1}{n+1}\right)^n \leq h(x) \leq \left(1 + \frac{1}{n}\right)^{n+1}. \quad \text{(Verify!)} \tag{8}$$

As $x \to +\infty$, n tends to $+\infty$ over *integers*, and by rules for *sequences*,

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n+1} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n}\right)^n = 1 \cdot \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 1 \cdot e = e,$$

with e as in Chapter 3, §15. Similarly one shows that also

$$\lim_{n \to \infty} \left(1 + \frac{1}{n+1}\right)^n = e.$$

Thus (8) implies that also $\lim_{x \to +\infty} h(x) = e$ (see Problem 6 below).

Remark. Here we used Corollary 2(ii) with

$$f(x) = [x], \; q = +\infty, \text{ and } g(n) = \left(1 + \frac{1}{n}\right)^n.$$

The substitution $n = f(x)$ is admissible since $f(x) = n$ never *equals* $+\infty$, its limit, thus satisfying Corollary 2(ii).

(B) Quite similarly, one shows that also

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

See Problem 5.

[3] In particular, the so-called *linear* substitution $y = ax + b$ ($a, b \in E^1$, $a \neq 0$) is always admissible since $f(x) = ax + b$ yields a *one-to-one* map.

(C) In Examples (A) and (B), we now substitute $x = 1/z$. This is admissible by Corollary 2(ii) since the dependence between x and z is one to one. Then

$$z = \frac{1}{x} \to 0^+ \text{ as } x \to +\infty, \text{ and } z \to 0^- \text{ as } x \to -\infty.$$

Thus (A) and (B) yield

$$\lim_{z\to 0^+}(1+z)^{1/z} = \lim_{z\to 0^-}(1+z)^{1/z} = e.$$

Hence by Corollary 3 of §1, we obtain

$$\lim_{z\to 0}(1+z)^{1/z} = e. \tag{9}$$

More Problems on Limits and Continuity

1. Complete the proof of Corollary 1.
[Hint: Consider $\{f(x_m)\}$ and $\{f(x'_m)\}$, with

$$x_m \to p \text{ and } x'_m \to p.$$

By Chapter 3, §14, Corollary 4, p is also the limit of

$$x_1, x'_1, x_2, x'_2, \ldots,$$

so, by assumption,

$$f(x_1), f(x'_1), \ldots \textit{ converges } (\text{to } q, \text{ say}).$$

Hence $\{f(x_m)\}$ and $\{f(x'_m)\}$ must have the *same* limit q. (Why?)]

***2.** Complete the converse proof of Theorem 2 (cf. proof of Theorem 1 in Chapter 3, §17).

3. Define $f, g\colon E^1 \to E^1$ by setting

(i) $f(x) = 2$; $g(y) = 3$ if $y \neq 2$, and $g(2) = 0$; or

(ii) $f(x) = 2$ if x is rational and $f(x) = 2x$ otherwise; g as in (i).

In both cases, show that

$$\lim_{x\to 1} f(x) = 2 \text{ and } \lim_{y\to 2} g(y) = 3 \text{ but } \textit{not } \lim_{x\to 1} g(f(x)) = 3.^4$$

4. Prove Theorem 3 from "ε, δ" definitions. Also prove (*both* ways) that if f is relatively continuous on B, and g on $f[B]$, then $g \circ f$ is relatively continuous on B.

[4] In case (ii), disprove the *existence* of $\lim_{x\to 1} g(f(x))$.

5. Complete the missing details in Examples (A) and (B).
[Hint for (B): Verify that

$$\left(1-\frac{1}{n+1}\right)^{-n-1}=\left(\frac{n}{n+1}\right)^{-n-1}=\left(\frac{n+1}{n}\right)^{n+1}=\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{n}\to e.]$$

⇒**6.** Given $f, g, h\colon A \to E^*$, $A \subseteq (S, \rho)$, with

$$f(x) \le h(x) \le g(x)$$

for $x \in G_{\neg p}(\delta) \cap A$ for some $\delta > 0$. Prove that if

$$\lim_{x\to p} f(x) = \lim_{x\to p} g(x) = q,$$

then also

$$\lim_{x\to p} h(x) = q.$$

Use Theorem 1.
[Hint: Take any

$$\{x_m\} \subseteq A - \{p\} \text{ with } x_m \to p.$$

Then $f(x_m) \to q$, $g(x_m) \to q$, and

$$(\forall\, x_m \in A \cap G_{\neg p}(\delta)) \quad f(x_m) \le h(x_m) \le g(x_m).$$

Now apply Corollary 3 of Chapter 3, §15.]

⇒**7.** Given $f, g\colon A \to E^*$, $A \subseteq (S, \rho)$, with $f(x) \to q$ and $g(x) \to r$ as $x \to p$ $(p \in S)$, prove the following:

(i) If $q > r$, then

$$(\exists\, \delta > 0)\ (\forall\, x \in A \cap G_{\neg p}(\delta)) \quad f(x) > g(x).$$

(ii) (*Passage to the limit in inequalities.*) If

$$(\forall\, \delta > 0)\ (\exists\, x \in A \cap G_{\neg p}(\delta)) \quad f(x) \le g(x),$$

then $q \le r$. (Observe that here A *clusters* at p necessarily, so the limits are *unique*.)
[Hint: Proceed as in Problem 6; use Corollary 1 of Chapter 3, §15.]

8. Do Problems 6 and 7 using only Definition 2 of §1.
[Hint: Here prove 7(ii) *first*.]

9. Do Examples (a)–(d) of §1 *using Theorem* 1.
[Hint: For (c), use also Example (a) in Chapter 3, §16.]

10. Addition and multiplication in E^1 may be treated as functions

$$f, g\colon E^2 \to E^1$$

with

$$f(x, y) = x + y \text{ and } g(x, y) = xy.$$

Show that f and g are continuous *on* E^2 (see footnote 2 in Chapter 3, §15). Similarly, show that the *standard metric*

$$\rho(x, y) = |x - y|$$

is a continuous mapping from E^2 to E^1.

[Hint: Use Theorems 1, 2, and, 4 of Chapter 3, §15 and the sequential criterion.]

11. Using Corollary 2 and formula (9), find $\lim_{x\to 0}(1 \pm mx)^{1/x}$ for a fixed $m \in N$.

⇒12. Let $a > 0$ in E^1. Prove that $\lim_{x\to 0} a^x = 1$.

[Hint: Let $n = f(x)$ be the integral part of $\frac{1}{x}$ $(x \neq 0)$. Verify that

$$a^{-1/(n+1)} \le a^x \le a^{1/n} \text{ if } a \ge 1,$$

with inequalities reversed if $0 < a < 1$. Then proceed as in Example (A), noting that

$$\lim_{n\to\infty} a^{1/n} = 1 = \lim_{n\to\infty} a^{-1/(n+1)}$$

by Problem 20 of Chapter 3, §15. (Explain!)]

⇒13. Given $f, g\colon A \to E^*$, $A \subseteq (S, \rho)$, with

$$f \le g \quad \text{for } x \text{ in } G_{\neg p}(\delta) \cap A.$$

Prove that

(a) if $\lim_{x\to p} f(x) = +\infty$, then also $\lim_{x\to p} g(x) = +\infty$;

(b) if $\lim_{x\to p} g(x) = -\infty$, then also $\lim_{x\to p} f(x) = -\infty$.

Do it it two ways:

(i) Use definitions only, such as (2′) in §1.

(ii) Use Problem 10 of Chapter 2, §13 and the sequential criterion.

⇒14. Prove that

(i) if $a > 1$ in E^1, then

$$\lim_{x\to+\infty} \frac{a^x}{x} = +\infty \text{ and } \lim_{x\to+\infty} \frac{a^{-x}}{x} = 0;$$

(ii) if $0 < a < 1$, then

$$\lim_{x\to+\infty} \frac{a^x}{x} = 0 \text{ and } \lim_{x\to+\infty} \frac{a^{-x}}{x} = +\infty;$$

(iii) if $a > 1$ and $0 \le q \in E^1$, then

$$\lim_{x\to+\infty} \frac{a^x}{x^q} = +\infty \text{ and } \lim_{x\to+\infty} \frac{a^{-x}}{x^q} = 0;$$

(iv) if $0 < a < 1$ and $0 \le q \in E^1$, then

$$\lim_{x\to+\infty} \frac{a^x}{x^q} = 0 \text{ and } \lim_{x\to+\infty} \frac{a^{-x}}{x^q} = +\infty.$$

[Hint: (i) From Problems 17 and 10 of Chapter 3, §15, obtain

$$\lim \frac{a^n}{n} = +\infty.$$

Then proceed as in Examples (A)–(C); (iii) reduces to (i) by the method used in Problem 18 of Chapter 3, §15.]

⇒***15.** For a map $f\colon (S,\,\rho) \to (T,\,\rho')$, show that the following statements are equivalent:

(i) f is continuous on S.

(ii) $(\forall\, A \subseteq S)\ f[\overline{A}] \subseteq \overline{f[A]}$.

(iii) $(\forall\, B \subseteq T)\ f^{-1}[\overline{B}] \supseteq \overline{f^{-1}[B]}$.

(iv) $f^{-1}[B]$ is closed in $(S,\,\rho)$ whenever B is closed in $(T,\,\rho')$.

(v) $f^{-1}[B]$ is open in $(S,\,\rho)$ whenever B is open in $(T,\,\rho')$.

[Hints: (i) $\Longrightarrow$ (ii): Use Theorem 3 of Chapter 3, §16 and the sequential criterion to show that

$$p \in \overline{A} \Longrightarrow f(p) \in \overline{f[A]}.$$

(ii) $\Longrightarrow$ (iii): Let $A = f^{-1}[B]$. Then $f[A] \subseteq B$, so by (ii),

$$f[\overline{A}] \subseteq \overline{f[A]} \subseteq \overline{B}.$$

Hence

$$\overline{f^{-1}[B]} = \overline{A} \subseteq f^{-1}[f[\overline{A}]] \subseteq f^{-1}[\overline{B}]. \quad \text{(Why?)}$$

(iii) $\Longrightarrow$ (iv): If B is closed, $B = \overline{B}$ (Chapter 3, §16, Theorem 4(ii)), so by (iii),

$$f^{-1}[B] = f^{-1}[\overline{B}] \supseteq \overline{f^{-1}[B]}; \text{ deduce (iv).}$$

(iv) $\Longrightarrow$ (v): Pass to complements in (iv).
(v) $\Longrightarrow$ (i): Assume (v). Take any $p \in S$ and use Definition 1 in §1.]

16. Let $f\colon E^1 \to E^1$ be continuous. Define $g\colon E^1 \to E^2$ by

$$g(x) = (x,\, f(x)).$$

Prove that

(a) g and g^{-1} are one to one and continuous;

(b) the range of g, i.e., the set

$$D'_g = \{(x,\, f(x)) \mid x \in E^1\},$$

is closed in E^2.

[Hint: Use Theorem 2 of Chapter 3, §15, Theorem 4 of Chapter 3, §16, and the sequential criterion.]

§3. Operations on Limits. Rational Functions

I. A function $f\colon A \to T$ is said to be *real* if its range D'_f lies in E^1, *complex* if $D'_f \subseteq C$, *vector valued* if D'_f is a subset of E^n, and *scalar valued* if D'_f lies in the scalar field of E^n. (*In the latter two cases, we use the same terminology if E^n is replaced by some other (fixed) normed space under consideration.) The domain A may be arbitrary.

For such functions one can define various operations whenever they are defined for elements of their *ranges*, to which the function values $f(x)$ belong. Thus as in Chapter 3, §9, we define the functions $f \pm g$, fg, and f/g "pointwise," setting

$$(f \pm g)(x) = f(x) \pm g(x),\ (fg)(x) = f(x)\, g(x),\ \text{and}\ \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

whenever the right side expressions are defined. We also define $|f|\colon A \to E^1$ by

$$(\forall\, x \in A) \quad |f|(x) = |f(x)|.$$

In particular, $f \pm g$ is defined if f and g are both vector valued or both scalar valued, and fg is defined if f is vector valued while g is scalar valued; similarly for f/g. (However, the domain of f/g consists of those $x \in A$ only for which $g(x) \neq 0$.)

In the theorems below, all limits are at some (arbitrary, but fixed) point p of the domain space (S, ρ). For brevity, we often omit "$x \to p$."

Theorem 1. *For any functions $f, g, h\colon A \to E^1(C)$, $A \subseteq (S, \rho)$, we have the following:*

(i) *If f, g, h are continuous at p $(p \in A)$, so are $f \pm g$ and fh. So also is f/h, provided $h(p) \neq 0$; similarly for* relative *continuity over $B \subseteq A$.*

(ii) *If $f(x) \to q$, $g(x) \to r$, and $h(x) \to a$ (all, as $x \to p$ over $B \subseteq A$), then*

(a) $f(x) \pm g(x) \to q \pm r$;

(b) $f(x)h(x) \to qa$; *and*

(c) $\dfrac{f(x)}{h(x)} \to \dfrac{q}{a}$, *provided $a \neq 0$.*

All this holds also if f and g are vector valued and h is scalar valued.

For a simple proof, one can use Theorem 1 of Chapter 3, §15. (An independent proof is sketched in Problems 1–7 below.)

We can also use the sequential criterion (Theorem 1 in §2). To prove (ii), take any sequence

$$\{x_m\} \subseteq B - \{p\},\ x_m \to p.$$

Then by the assumptions made,

$$f(x_m) \to q,\ g(x_m) \to r, \text{ and } h(x_m) \to a.$$

Thus by Theorem 1 of Chapter 3, §15,

$$f(x_m) \pm g(x_m) \to q \pm r,\ f(x_m)g(x_m) \to qa, \text{ and } \frac{f(x_m)}{g(x_m)} \to \frac{q}{a}.$$

As this holds for *any* sequence $\{x_m\} \subseteq B - \{p\}$ with $x_m \to p$, our assertion (ii) follows by the sequential criterion; similarly for (i).

Note 1. By induction, the theorem also holds for sums and products of any *finite* number of functions (whenever such products are defined).

Note 2. Part (ii) *does not apply to infinite limits* q, r, a; but it does apply to limits *at* $p = \pm\infty$ (take E^* with a suitable metric for the space S).

Note 3. The assumption $h(x) \to a \neq 0$ (as $x \to p$ over B) implies that $h(x) \neq 0$ for x in $B \cap G_{\neg p}(\delta)$ for some $\delta > 0$; see Problem 5 below. Thus the quotient function f/h *is defined on* $B \cap G_{\neg p}(\delta)$ *at least*.

II. If the range space of f is E^n (*or C^n), then each function value $f(x)$ is a *vector* in that space; thus it has n real (*respectively, complex) components, denoted

$$f_k(x), \quad k = 1, 2, \ldots, n.$$

Here we may treat f_k as a mapping of $A = D_f$ *into* E^1 (* *or* C); it carries each point $x \in A$ into $f_k(x)$, the kth component of $f(x)$. In this manner, *each function*

$$f\colon A \to E^n\ (C^n)$$

uniquely determines n *scalar-valued maps*

$$f_k\colon A \to E^1\ (C),$$

called the *components* of f. Notation: $f = (f_1, \ldots, f_n)$.

Conversely, given n arbitrary functions

$$f_k : A \to E^1\ (C), \quad k = 1, 2, \ldots, n,$$

one can define $f\colon A \to E^n$ (C^n) by setting

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x)).$$

Then obviously $f = (f_1, f_2, \ldots, f_n)$. Thus the f_k in turn determine f uniquely. *To define a function* $f\colon A \to E^n$ (C^n) *means to give its* n *components* f_k. Note that

$$f(x) = (f_1(x), \ldots, f_n(x)) = \sum_{k=1}^{n} \bar{e}_k f_k(x), \quad \text{i.e., } f = \sum_{k=1}^{n} \bar{e}_k f_k, \tag{1}$$

where the $\bar{e}_k$ are the n basic unit vectors; see Chapter 3, §§1–3, Theorem 2. Our next theorem shows that the limits and continuity of f reduce to those of the f_k.

Theorem 2 (componentwise continuity and limits)**.** *For any function $f\colon A \to E^n$ (C^n), with $A \subseteq (S, \rho)$ and with $f = (f_1, \ldots, f_n)$, we have that*

(i) *f is continuous at p $(p \in A)$ iff all its components f_k are, and*

(ii) *$f(x) \to \bar{q}$ as $x \to p$ $(p \in S)$ iff*

$$f_k(x) \to q_k \text{ as } x \to p \quad (k = 1, 2, \ldots, n),$$

i.e., iff each f_k has, as its limit at p, the corresponding component of $\bar{q}$.

Similar results hold for relative *continuity and limits over a path $B \subseteq A$.*

We prove (ii). If $f(x) \to \bar{q}$ as $x \to p$ then, by definition,

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, x \in A \cap G_{\neg p}(\delta)) \quad \varepsilon > |f(x) - \bar{q}| = \sqrt{\sum_{k=1}^{n} |f_k(x) - q_k|^2}\,;$$

in turn, the right-hand side of the inequality given above is no less than each

$$|f_k(x) - q_k|, \quad k = 1, 2, \ldots, n.$$

Thus

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, x \in A \cap G_{\neg p}(\delta)) \quad |f_k(x) - q_k| < \varepsilon;$$

i.e., $f_k(x) \to q_k$, $k = 1, \ldots, n$.

Conversely, if each $f_k(x) \to q_k$, then Theorem 1(ii) yields

$$\sum_{k=1}^{n} \bar{e}_k f_k(x) \to \sum_{k=1}^{n} \bar{e}_k q_k.$$ [1]

By formula (1), then, $f(x) \to \bar{q}$ (for $\sum_{k=1}^{n} \bar{e}_k q_k = \bar{q}$). Thus (ii) is proved; similarly for (i) and for relative limits and continuity.

Note 4. Again, Theorem 2 holds also for $p = \pm\infty$ (but not for infinite q).

Note 5. A *complex* function $f\colon A \to C$ may be treated as $f\colon A \to E^2$. Thus it has two *real* components: $f = (f_1, f_2)$. Traditionally, f_1 and f_2 are called the *real* and *imaginary* parts of f, also denoted by f_{re} and f_{im}, so

$$f = f_{\text{re}} + i \cdot f_{\text{im}}.$$

By Theorem 2, *f is continuous at p iff f_{re} and f_{im} are.*

[1] Here we treat $\bar{e}_k$ as a *constant function*, with values $\bar{e}_k$ (cf. §1, Example (a)).

Example.

The *complex exponential* is the function $f\colon E^1 \to C$ defined by

$$f(x) = \cos x + i \cdot \sin x, \text{ also written } f(x) = e^{xi}.$$

As we shall see later, the sine and the cosine functions are continuous. Hence so is f by Theorem 2.

III. Next, consider functions whose *domain* is a set in E^n (*or C^n). We call them *functions of* n *real* (* *or complex*) *variables*, treating $\bar{x} = (x_1, \ldots, x_n)$ as a variable n-tuple. The *range* space may be arbitrary.

In particular, a *monomial* in n variables is a map on E^n (*or C^n) given by a formula of the form

$$f(\bar{x}) = ax_1^{m_1}x_2^{m_2}\cdots x_n^{m_n} = a \cdot \prod_{k=1}^{n} x_k^{m_k},$$

where the m_k are fixed integers ≥ 0 and $a \in E^1$ (*or $a \in C$).[2] If $a \neq 0$, the sum $m = \sum_{k=1}^{n} m_k$ is called the *degree* of the monomial. Thus

$$f(x, y, z) = 3x^2yz^3 = 3x^2y^1z^3$$

defines a monomial of degree 6, in three real (or complex) variables x, y, z. (We often write x, y, z for x_1, x_2, x_3.)

A *polynomial* is any sum of a finite number of monomials; its degree is, by definition, that of its *leading term*, i.e., the one of highest degree. (There may be several such terms, of equal degree.) For example,

$$f(x, y, z) = 3x^2yz^3 - 2xy^7$$

defines a polynomial of degree 8 in x, y, z. Polynomials of degree 1 are sometimes called *linear*.

A *rational function* is the quotient f/g of two polynomials f and g on E^n (*or C^n).[3] Its domain consists of those points at which g does not vanish. For example,

$$h(x, y) = \frac{x^2 - 3xy}{xy - 1}$$

defines a rational function on points (x, y), with $xy \neq 1$. Polynomials and monomials are rational functions with denominator 1.

Theorem 3. *Any rational function* (*in particular*, *every polynomial*) *in one or several variables is continuous on all of its domain.*

[2] We also allow a to be a *vector*, while the x_k are *scalars*.

[3] This is valid also if one allows the coefficients of f to be *vectors* (provided those of g, and the variables x_k, remain scalars).

Proof. Consider first a monomial of the form

$$f(\bar{x}) = x_k \quad (k \text{ fixed});$$

it is called the *kth projection map* because it "projects" each $\bar{x} \in E^n$ ($^*C^n$) onto its kth component x_k.

Given any $\varepsilon > 0$ and $\bar{p}$, choose $\delta = \varepsilon$. Then

$$(\forall\, \bar{x} \in G_{\bar{p}}(\delta)) \quad |f(\bar{x}) - f(\bar{p})| = |x_k - p_k| \le \sqrt{\sum_{i=1}^{n} |x_i - p_i|^2} = \rho(\bar{x}, \bar{p}) < \varepsilon.$$

Hence by definition, f is continuous at each $\bar{p}$. Thus the theorem holds for projection maps.

However, any other monomial, given by

$$f(\bar{x}) = a x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n},$$

is the product of finitely many (namely of $m = m_1 + m_2 + \cdots + m_n$) projection maps multiplied by a constant a. Thus by Theorem 1, it is continuous. So also is any finite sum of monomials (i.e., any polynomial), and hence so is the quotient f/g of two polynomials (i.e., any rational function) wherever it is defined, i.e., wherever the denominator does not vanish. $\square$

IV. For functions on E^n (*or C^n), we often consider relative limits *over a line of the form*

$$\bar{x} = \bar{p} + t\vec{e}_k \text{ (parallel to the } k\text{th axis, through } \bar{p});$$

see Chapter 3, §§4–6, Definition 1. If f is relatively continuous at $\bar{p}$ *over that line*, we say that f is continuous at $\bar{p}$ *in the kth variable* x_k (because the other components of $\bar{x}$ remain *constant*, namely, equal to those of $\bar{p}$, as $\bar{x}$ runs over that line). As opposed to this, we say that f is continuous at $\bar{p}$ *in all n variables jointly* if it is continuous at $\bar{p}$ in the ordinary (not relative) sense. Similarly, we speak of *limits* in one variable, or in all of them jointly.

Since ordinary continuity implies relative continuity over *any* path, joint *continuity in all n variables always implies that in each variable separately*, but *the converse fails* (see Problems 9 and 10 below); similarly for *limits* at $\bar{p}$.

Problems on Continuity of Vector-Valued Functions

1. Give an "ε, δ" proof of Theorem 1 for $f \pm g$.

[Hint: Proceed as in Theorem 1 of Chapter 3, §15, replacing $\max(k', k'')$ by $\delta = \min(\delta', \delta'')$. Thus fix $\varepsilon > 0$ and $p \in S$. If $f(x) \to q$ and $g(x) \to r$ as $x \to p$ over B, then $(\exists\, \delta', \delta'' > 0)$ such that

$$(\forall\, x \in B \cap G_{\neg p}(\delta')) \quad |f(x) - q| < \frac{\varepsilon}{2} \text{ and } (\forall\, x \in B \cap G_{\neg p}(\delta'')) \quad |g(x) - r| < \frac{\varepsilon}{2}.$$

Put $\delta = \min(\delta', \delta'')$, etc.]

In Problems 2, 3, and 4, $E = E^n$ (*or another normed space), F is its scalar field, $B \subseteq A \subseteq (S, \rho)$, and $x \to p$ *over* B.

2. For a function $f\colon A \to E$ prove that

$$f(x) \to q \iff |f(x) - q| \to 0,$$

equivalently, iff $f(x) - q \to \bar{0}$.

[Hint: Proceed as in Chapter 3, §14, Corollary 2.]

3. Given $f\colon A \to (T, \rho')$, with $f(x) \to q$ as $x \to p$ over B. Show that for some $\delta > 0$, f *is bounded on* $B \cap G_{\neg p}(\delta)$, i.e.,

$$f[B \cap G_{\neg p}(\delta)] \text{ is a bounded set in } (T, \rho').$$

Thus if $T = E$, there is $K \in E^1$ such that

$$(\forall\, x \in B \cap G_{\neg p}(\delta)) \quad |f(x)| < K$$

(Chapter 3, §13, Theorem 2).

4. Given $f, h\colon A \to E^1$ (C) (or $f\colon A \to E$, $h\colon A \to F$), prove that if one of f and h has limit 0 (respectively, $\bar{0}$), while the other is bounded on $B \cap G_{\neg p}(\delta)$, then $h(x)f(x) \to 0$ $(\bar{0})$.

5. Given $h\colon A \to E^1$ (C), with $h(x) \to a$ as $x \to p$ over B, and $a \neq 0$. Prove that

$$(\exists\, \varepsilon, \delta > 0)\ (\forall\, x \in B \cap G_{\neg p}(\delta)) \quad |h(x)| \geq \varepsilon,$$

i.e., $h(x)$ is *bounded away from* 0 on $B \cap G_{\neg p}(\delta)$. Hence show that $1/h$ is *bounded* on $B \cap G_{\neg p}(\delta)$.

[Hint: Proceed as in the proof of Corollary 1 in §1, with $q = a$ and $r = 0$. Then use

$$(\forall\, x \in B \cap G_{\neg p}(\delta)) \quad \left|\frac{1}{h(x)}\right| \leq \frac{1}{\varepsilon}.]$$

6. Using Problems 1 to 5, give an independent proof of Theorem 1.

[Hint: Proceed as in Problems 2 and 4 of Chapter 3, §15 to obtain Theorem 1(ii). Then use Corollary 2 of §1.]

7. Deduce Theorems 1 and 2 of Chapter 3, §15 from those of the present section, setting $A = B = N$, $S = E^*$, and $p = +\infty$.

[Hint: See §1, Note 5.]

8. Redo Problem 8 of §1 in two ways:

(i) Use Theorem 1 only.

(ii) Use Theorem 3.

[Example for (i): Find $\lim_{x\to 1}(x^2 + 1)$.

Here $f(x) = x^2 + 1$, or $f = gg + h$, where $h(x) = 1$ (constant) and $g(x) = x$ (identity map). As h and g are continuous (§1, Examples (a) and (b)), so is f by Theorem 1. Thus $\lim_{x\to 1} f(x) = f(1) = 1^2 + 1 = 2$.

Or, using Theorem 1(ii), $\lim_{x\to 1}(x^2+1) = \lim_{x\to 1} x^2 + \lim_{x\to 1} 1$, etc.]

9. Define $f\colon E^2 \to E^1$ by

$$f(x,\, y) = \frac{x^2 y}{(x^4+y^2)}, \text{ with } f(0,0)=0.$$

Show that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along any straight line through $\bar{0}$, but *not* over the parabola $y = x^2$ (then the limit is $\frac{1}{2}$). Deduce that f is continuous at $\bar{0} = (0, 0)$ in x and y *separately*, but not *jointly*.

10. Do Problem 9, setting

$$f(x,\, y) = 0 \text{ if } x = 0, \text{ and } f(x,\, y) = \frac{|y|}{x^2} \cdot 2^{-|y|/x^2} \text{ if } x \neq 0.^4$$

11. Discuss the continuity of $f\colon E^2 \to E^1$ in x and y jointly and separately, at $\bar{0}$, when

(a) $f(x,\, y) = \dfrac{x^2y^2}{x^2+y^2}$, $f(0,\, 0) = 0$;

(b) $f(x,\, y) =$ integral part of $x + y$;

(c) $f(x,\, y) = x + \dfrac{xy}{|x|}$ if $x \neq 0$, $f(0,\, y) = 0$;

(d) $f(x,\, y) = \dfrac{xy}{|x|} + x \sin\dfrac{1}{y}$ if $xy \neq 0$, and $f(x,\, y) = 0$ otherwise;

(e) $f(x,\, y) = \dfrac{1}{x}\sin(x^2 + |xy|)$ if $x \neq 0$, and $f(0,\, y) = 0$.

[Hints: In (c) and (d), $|f(x,\, y)| \leq |x| + |y|$; in (e), use $|\sin\alpha| \leq |\alpha|$.]

[4] Use Problem 14 in §2 for limit computations.

§4. Infinite Limits. Operations in E^*

As we have noted, Theorem 1 of §3 does not apply to *infinite limits*,[1] even if the function *values* $f(x)$, $g(x)$, $h(x)$ remain finite (i.e., *in* E^1). Only in certain cases (stated below) can we prove some analogues.

There are quite a few such separate cases. Thus, for brevity, we shall adopt a kind of mathematical shorthand. The letter q will not necessarily denote a *constant*; it will stand for

"a function $f\colon A \to E^1$, $A \subseteq (S, \rho)$, such that $f(x) \to q \in E^1$ as $x \to p$."[2]

Similarly, "0" and "$\pm\infty$" will stand for analogous expressions, with q replaced by 0 and $\pm\infty$, respectively.

For example, the "shorthand formula" $(+\infty) + (+\infty) = +\infty$ means

> "The sum of two real functions, with limit $+\infty$ at p ($p \in S$), is itself a function with limit $+\infty$ at p."[3]

The point p is fixed, possibly $\pm\infty$ (if $A \subseteq E^*$). With this notation, we have the following theorems.

Theorems.

1. $(\pm\infty) + (\pm\infty) = \pm\infty.$

2. $(\pm\infty) + q = q + (\pm\infty) = \pm\infty.$

3. $(\pm\infty) \cdot (\pm\infty) = +\infty.$

4. $(\pm\infty) \cdot (\mp\infty) = -\infty.$

5. $|\pm\infty| = +\infty.$

6. $(\pm\infty) \cdot q = q \cdot (\pm\infty) = \pm\infty$ *if* $q > 0.$

7. $(\pm\infty) \cdot q = q \cdot (\pm\infty) = \mp\infty$ *if* $q < 0.$

8. $-(\pm\infty) = \mp\infty.$

9. $\dfrac{(\pm\infty)}{q} = (\pm\infty) \cdot \dfrac{1}{q}$ *if* $q \neq 0.$

10. $\dfrac{q}{(\pm\infty)} = 0.$

11. $(+\infty)^{+\infty} = +\infty.$

12. $(+\infty)^{-\infty} = 0.$

13. $(+\infty)^q = +\infty$ *if* $q > 0.$

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[1] It even has no meaning since operations on $\pm\infty$ have not been defined.

[2] Note that q is *finite* throughout.

[3] Similarly for $(-\infty) + (-\infty) = -\infty$. Both combined are written as "$(\pm\infty) + (\pm\infty) = \pm\infty$."

14. $(+\infty)^q = 0$ *if* $q < 0$.

15. *If* $q > 1$, *then* $q^{+\infty} = +\infty$ *and* $q^{-\infty} = 0$.

16. *If* $0 < q < 1$, *then* $q^{+\infty} = 0$ *and* $q^{-\infty} = +\infty$.

We prove Theorems 1 and 2, leaving the rest as problems. (Theorems 11–16 are best postponed until the theory of logarithms is developed.)

1. Let $f(x)$ and $g(x) \to +\infty$ as $x \to p$. We have to show that

$$f(x) + g(x) \to +\infty,$$

i.e., that

$$(\forall\, b \in E^1)\ (\exists\, \delta > 0)\ (\forall\, x \in A \cap G_{\neg p}(\delta)) \quad f(x) + g(x) > b$$

(we may assume $b > 0$). Thus fix $b > 0$. As $f(x)$ and $g(x) \to +\infty$, there are δ', $\delta'' > 0$ such that

$$(\forall\, x \in A \cap G_{\neg p}(\delta'))\ f(x) > b \text{ and } (\forall\, x \in A \cap G_{\neg p}(\delta''))\ g(x) > b.$$

Let $\delta = \min(\delta', \delta'')$. Then

$$(\forall\, x \in A \cap G_{\neg p}(\delta)) \quad f(x) + g(x) > b + b > b,$$

as required; similarly for the case of $-\infty$.

2. Let $f(x) \to +\infty$ and $g(x) \to q \in E^1$. Then there is $\delta' > 0$ such that for x in $A \cap G_{\neg p}(\delta')$, $|q - g(x)| < 1$, so that $g(x) > q - 1$.

Also, given any $b \in E^1$, there is δ'' such that

$$(\forall\, x \in A \cap G_{\neg p}(\delta'')) \quad f(x) > b - q + 1.$$

Let $\delta = \min(\delta', \delta'')$. Then

$$(\forall\, x \in A \cap G_{\neg p}(\delta)) \quad f(x) + g(x) > (b - q + 1) + (q - 1) = b,$$

as required; similarly for the case of $f(x) \to -\infty$.

Caution: No theorems of this kind exist for the following cases (which therefore are called *indeterminate expressions*):

$$(+\infty) + (-\infty), \quad (\pm\infty) \cdot 0, \quad \frac{\pm\infty}{\pm\infty}, \quad \frac{0}{0}, \quad (\pm\infty)^0, \quad 0^0, \quad 1^{\pm\infty}. \tag{1*}$$

In these cases, it does not suffice to know only the *limits* of f and g. It is necessary to investigate *the functions themselves* to give a definite answer, since in each case the answer may be different, depending on the properties of f and g. The expressions (1*) remain indeterminate even if we consider the simplest kind of functions, namely *sequences*, as we show next.

Examples.

(a) Let

$$u_m = 2m \text{ and } v_m = -m.$$

(This corresponds to $f(x) = 2x$ and $g(x) = -x$.) Then, as is readily seen,

$$u_m \to +\infty,\ v_m \to -\infty, \text{ and } u_m + v_m = 2m - m = m \to +\infty.$$

If, however, we take $x_m = 2m$ and $y_m = -2m$, then

$$x_m + y_m = 2m - 2m = 0;$$

thus $x_m + y_m$ is *constant*, with limit 0 (for the limit of a constant function equals its value; see §1, Example (a)).

Next, let

$$u_m = 2m \text{ and } z_m = -2m + (-1)^m.$$

Then again

$$u_m \to +\infty \text{ and } z_m \to -\infty, \text{ but } u_m + z_m = (-1)^m;$$

$u_m + z_m$ "oscillates" from -1 to 1 as $m \to +\infty$, so it has no limit at all.

These examples show that $(+\infty) + (-\infty)$ is indeed an indeterminate expression since the answer depends on the nature of the functions involved. No *general* answer is possible.

(b) We now show that $1^{+\infty}$ is indeterminate.

Take first a *constant* $\{x_m\}$, $x_m = 1$, and let $y_m = m$. Then

$$x_m \to 1,\ y_m \to +\infty, \text{ and } x_m^{y_m} = 1^m = 1 = x_m \to 1.$$

If, however, $x_m = 1 + \frac{1}{m}$ and $y_m = m$, then again $y_m \to +\infty$ and $x_m \to 1$ (by Theorem 10 above and Theorem 1 of Chapter 3, §15), but

$$x_m^{y_m} = \left(1 + \frac{1}{m}\right)^m$$

does *not* tend to 1; it tends to $e > 2$, as shown in Chapter 3, §15. Thus again the result depends on $\{x_m\}$ and $\{y_m\}$.

In a similar manner, one shows that the other cases (1^*) are indeterminate.

Note 1. It is often useful to introduce additional "shorthand" conventions. Thus the symbol ∞ (*unsigned infinity*) might denote a function f such that

$$|f(x)| \to +\infty \text{ as } x \to p;$$

we then also write $f(x) \to \infty$. The symbol 0^+ (respectively, 0^-) denotes a function f such that

$$f(x) \to 0 \text{ as } x \to p$$

and, *moreover*,

$$f(x) > 0 \ (f(x) < 0, \textit{ respectively}) \textit{ on some } G_{\neg p}(\delta).$$

We then have the following additional formulas:

(i) $\frac{(\pm\infty)}{0^+} = \pm\infty$, $\frac{(\pm\infty)}{0^-} = \mp\infty$.

(ii) If $q > 0$, then $\frac{q}{0^+} = +\infty$ and $\frac{q}{0^-} = -\infty$.

(iii) $\frac{\infty}{0} = \infty$.

(iv) $\frac{q}{\infty} = 0$.

The proof is left to the reader.

Note 2. All these formulas and theorems hold for relative limits, too.

So far, we have defined no arithmetic operations in E^*. To fill this gap (at least partially), *we shall henceforth treat Theorems* 1–16 *above not only as certain limit statements* (*in* "*shorthand*") *but also as definitions* of certain operations in E^*. For example, the formula $(+\infty)+(+\infty) = +\infty$ shall be treated as the definition of the *actual sum* of $+\infty$ and $+\infty$ in E^*, with $+\infty$ regarded this time as an *element of* E^* (not as a function). This convention defines the arithmetic operations for certain cases only; the indeterminate expressions (1^*) remain undefined, unless we decide to assign them some meaning.

In higher analysis, it indeed proves convenient to assign a meaning to at least some of them. We shall adopt these (admittedly arbitrary) conventions:

$$\begin{cases} (\pm\infty) + (\mp\infty) = (\pm\infty) - (\pm\infty) = +\infty; \ 0^0 = 1; \\ 0 \cdot (\pm\infty) = (\pm\infty) \cdot 0 = 0 \text{ (even if 0 stands for the zero-}\textit{vector}\text{)}. \end{cases} \tag{2*}$$

Caution: These formulas must not be treated as limit theorems (in "shorthand"). Sums and products of the form (2^*) will be called "*unorthodox*."

Problems on Limits and Operations in E^*

1. Show by examples that all expressions (1^*) are indeterminate.

2. Give explicit definitions for the following "unsigned infinity" limit statements:

(a) $\lim_{x\to p} f(x) = \infty$; (b) $\lim_{x\to p^+} f(x) = \infty$; (c) $\lim_{x\to\infty} f(x) = \infty$.

3. Prove at least some of Theorems 1–10 and formulas (i)–(iv) in Note 1.

4. In the following cases, find $\lim f(x)$ in two ways: (i) use definitions only; (ii) use suitable theorems and justify each step accordingly.

(a) $\lim\limits_{x\to\infty} \dfrac{1}{x}$ $(=0)$. (b) $\lim\limits_{x\to\infty} \dfrac{x(x-1)}{1-3x^2}$.

(c) $\lim\limits_{x\to 2^+} \dfrac{x^2-2x+1}{x^2-3x+2}$. (d) $\lim\limits_{x\to 2^-} \dfrac{x^2-2x+1}{x^2-3x+2}$.

(e) $\lim\limits_{x\to 2} \dfrac{x^2-2x+1}{x^2-3x+2}$ $(=\infty)$.

[Hint: Before using theorems, reduce by a suitable power of x.]

5. Let

$$f(x)=\sum_{k=0}^{n} a_k x^k \text{ and } g(x)=\sum_{k=0}^{m} b_k x^k \ (a_n \neq 0,\ b_m \neq 0).$$

Find $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)}$ if (i) $n>m$; (ii) $n<m$; and (iii) $n=m$ $(n, m \in N)$.

6. Verify commutativity and associativity of addition and multiplication in E^*, treating Theorems 1–16 and formulas (2^*) as definitions. Show by examples that associativity and commutativity (for three terms or more) would fail if, instead of (2^*), the formula $(\pm\infty)+(\mp\infty)=0$ were adopted.

[Hint: For sums, first suppose that one of the terms in a sum is $+\infty$; then *the sum is* $+\infty$. For products, single out the case where one of the factors is 0; then consider the infinite cases.]

7. Continuing Problem 6, verify the distributive law $(x+y)z = xz+yz$ in E^*, assuming that x and y have *the same sign* (if infinite), or that $z \geq 0$. Show by examples that it may fail in other cases; e.g., if $x=-y=+\infty$, $z=-1$.

§5. Monotone Functions

A function $f\colon A \to E^*$, with $A \subseteq E^*$, is said to be *nondecreasing* on a set $B \subseteq A$ iff

$$x \leq y \text{ implies } f(x) \leq f(y) \text{ for } x, y \in B.$$

It is said to be *nonincreasing* on B iff

$$x \leq y \text{ implies } f(x) \geq f(y) \text{ for } x, y \in B.$$

Notation: $f\uparrow$ and $f\downarrow$ (on B), respectively.

In both cases, f is said to be *monotone* or *monotonic* on B. If f is also one to one *on* B (i.e., when restricted to B), we say that it is *strictly monotone* (*increasing* if $f\uparrow$ and *decreasing* if $f\downarrow$).

Clearly, f is nondecreasing iff the function $-f = (-1)f$ is nonincreasing. Thus in proofs, we need consider only the case $f\uparrow$. The case $f\downarrow$ reduces to it by applying the result to $-f$.

Theorem 1. *If a function $f\colon A \to E^*$ $(A \subseteq E^*)$ is monotone on A, it has a left and a right (possibly infinite) limit at each point $p \in E^*$.*

In particular, if $f\uparrow$ on an interval $(a, b) \neq \emptyset$, then

$$f(p^-) = \sup_{a<x<p} f(x) \text{ for } p \in (a, b]$$

and

$$f(p^+) = \inf_{p<x<b} f(x) \text{ for } p \in [a, b).$$

(*In case $f\downarrow$, interchange* "sup" *and* "inf.")

Proof. To fix ideas, assume $f\uparrow$.

Let $p \in E^*$ and $B = \{x \in A \mid x < p\}$. Put $q = \sup f[B]$ (this sup *always* exists in E^*; see Chapter 2, §13). We shall show that q is a *left limit of f at p* (i.e., *a left limit over B*).

There are three possible cases:

(1) If q is *finite*, any globe G_q is an interval (c, d), $c < q < d$, in E^1. As $c < q = \sup f[B]$, *c cannot be an upper bound of $f[B]$* (why?), so c is exceeded by some $f(x_0)$, $x_0 \in B$. Thus

$$c < f(x_0),\ x_0 < p.$$

Hence as $f\uparrow$, we certainly have

$$c < f(x_0) \le f(x) \text{ for all } x > x_0 \ (x \in B).$$

Moreover, as $f(x) \in f[B]$, we have

$$f(x) \le \sup f[B] = q < d,$$

so $c < f(x) < d$; i.e., $f(x) \in (c, d) = G_q$.

We have thus shown that

$$(\forall\, G_q)\ (\exists\, x_0 < p)\ (\forall\, x \in B \mid x_0 < x) \quad f(x) \in G_q,$$

so q is a left limit at p.

(2) If $q = +\infty$, the same proof works with $G_q = (c, +\infty]$. Verify!

(3) If $q = -\infty$, then

$$(\forall\, x \in B) \quad f(x) \le \sup f[B] = -\infty,$$

i.e., $f(x) \le -\infty$, so $f(x) = -\infty$ (constant) on B. Hence q is also a left limit at p (§1, Example (a)).

In particular, if $f\uparrow$ on $A = (a, b)$ with $a, b \in E^*$ and $a < b$, then $B = (a, p)$ for $p \in (a, b]$. Here p is a *cluster point* of the path B (Chapter 3, §14, Example (h)), so a *unique* left limit $f(p^-)$ exists. By what was shown above,

$$q = f(p^-) = \sup f[B] = \sup_{a<x<p} f(x), \text{ as claimed.}$$

Thus all is proved for *left* limits.

The proof for *right* limits is quite similar; one only has to set

$$B = \{x \in A \mid x > p\},\ q = \inf f[B]. \quad \square$$

Note 1. The second clause of Theorem 1 holds even if (a, b) is only a subset of A, for the limits in question are not affected by *restricting* f to (a, b). (Why?) The endpoints a and b may be finite or infinite.

Note 2. If $D_f = A = N$ (the naturals), then by definition, $f\colon N \to E^*$ is a *sequence* with general term $x_m = f(m)$, $m \in N$ (see §1, Note 2). Then setting $p = +\infty$ in the proof of Theorem 1, we obtain Theorem 3 of Chapter 3, §15. (Verify!)

Example.

The *exponential function* $F : E^1 \to E^1$ to the base $a > 0$ is given by

$$F(x) = a^x.$$

It is monotone (Chapter 2, §§11–12, formula (1)), so $F(0^-)$ and $F(0^+)$ *exist*. By the sequential criterion (Theorem 1 of §2), we may use a suitable sequence to find $F(0^+)$, and we choose $x_m = \frac{1}{m} \to 0^+$. Then

$$F(0^+) = \lim_{m\to\infty} F\left(\frac{1}{m}\right) = \lim_{m\to\infty} a^{1/m} = 1$$

(see Chapter 3, §15, Problem 20).

Similarly, taking $x_m = -\frac{1}{m} \to 0^-$, we obtain $F(0^-) = 1$. Thus

$$F(0^+) = F(0^-) = \lim_{x\to 0} F(x) = \lim_{x\to 0} a^x = 1.$$

(See also Problem 12 of §2.)

Next, fix any $p \in E^1$. Noting that

$$F(x) = a^x = a^{p+x-p} = a^p a^{x-p},$$

we set $y = x - p$. (Why is this substitution admissible?) Then $y \to 0$ as $x \to p$, so we get

$$\lim_{x\to p} F(x) = \lim a^p \cdot \lim_{x\to p} a^{x-p} = a^p \lim_{y\to 0} a^y = a^p \cdot 1 = a^p = F(p).$$

As $\lim_{x\to p} F(x) = F(p)$, *$F$ is continuous at each $p \in E^1$.* Thus *all exponentials are continuous.*

Theorem 2. *If a function $f\colon A \to E^*$ $(A \subseteq E^*)$ is nondecreasing on a finite or infinite interval $B = (a, b) \subseteq A$ and if $p \in (a, b)$, then*

$$f(a^+) \le f(p^-) \le f(p) \le f(p^+) \le f(b^-), \tag{1}$$

and for no $x \in (a, b)$ *do we have*

$$f(p^-) < f(x) < f(p) \text{ or } f(p) < f(x) < f(p^+);$$[1]

similarly in case $f\downarrow$ (with all inequalities reversed).

Proof. By Theorem 1, $f\uparrow$ on (a, p) implies

$$f(a^+) = \inf_{a<x<p} f(x) \text{ and } f(p^-) = \sup_{a<x<p} f(x);$$

thus certainly $f(a^+) \le f(p^-)$. As $f\uparrow$, we also have $f(p) \ge f(x)$ for all $x \in (a, p)$; hence

$$f(p) \ge \sup_{a<x<p} f(x) = f(p^-).$$

Thus

$$f(a^+) \le f(p^-) \le f(p);$$

similarly for the rest of (1).

Moreover, if $a < x < p$, then $f(x) \le f(p^-)$ since

$$f(p^-) = \sup_{a<x<p} f(x).$$

If, however, $p \le x < b$, then $f(p) \le f(x)$ since $f\uparrow$. Thus *we never have* $f(p^-) < f(x) < f(p)$. Similarly, one excludes $f(p) < f(x) < f(p^+)$. This completes the proof. □

Note 3. If $f(p^-)$, $f(p^+)$, and $f(p)$ exist (all *finite*), then

$$|f(p) - f(p^-)| \text{ and } |f(p^+) - f(p)|$$

are called, respectively, the *left* and *right jumps* of f at p; their sum is the (*total*) *jump* at p. If f is monotone, the jump equals $|f(p^+) - f(p^-)|$.

For a graphical example, consider Figure 14 in §1. Here $f(p) = f(p^-)$ (both finite), so the left jump is 0. However, $f(p^+) > f(p)$, so the right jump is greater than 0. Since

$$f(p) = f(p^-) = \lim_{x\to p^-} f(x),$$

f is left continuous (but not right continuous) at p.

[1] In other words, the interval $[f(p^-),\, f(p^+)]$ contains *no* $f(x)$ *except* $f(p)$.

Theorem 3. *If $f: A \to E^*$ is monotone on a finite or infinite interval (a, b) contained in A, then all its discontinuities in (a, b), if any, are "jumps," that is, points p at which $f(p^-)$ and $f(p^+)$ exist, but $f(p^-) \neq f(p)$ or $f(p^+) \neq f(p)$.*[2]

Proof. By Theorem 1, $f(p^-)$ and $f(p^+)$ exist at each $p \in (a, b)$.

If, in addition, $f(p^-) = f(p^+) = f(p)$, then

$$\lim_{x \to p} f(x) = f(p)$$

by Corollary 3 of §1, so f is continuous at p. Thus discontinuities occur only if $f(p^-) \neq f(p)$ or $f(p^+) \neq f(p)$. □

Problems on Monotone Functions

1. Complete the proofs of Theorems 1 and 2. Give also an independent (analogous) proof for *nonincreasing* functions.

2. Discuss Examples (d) and (e) of §1 again using Theorems 1–3.

3. Show that Theorem 3 holds also if f is *piecewise monotone* on (a, b), i.e., monotone on each of a sequence of intervals whose union is (a, b).

4. Consider the monotone function f defined in Problems 5 and 6 of Chapter 3, §11. Show that under the standard metric in E^1, f is continuous on E^1 and f^{-1} is continuous on $(0, 1)$. Additionally, discuss continuity under the metric ρ'.

⇒**5.** Prove that if f is monotone on $(a, b) \subseteq E^*$, it has at most countably many discontinuities in (a, b).

[Hint: Let $f\uparrow$. By Theorem 3, all discontinuities of f correspond to mutually disjoint intervals $(f(p^-), f(p^+)) \neq \emptyset$. (Why?) Pick a *rational* from each such interval, so these rationals correspond one to one to the discontinuities and form a *countable* set (Chapter 1, §9)].

6. Continuing Problem 17 of Chapter 3, §14, let

$$G_{11} = \left(\frac{1}{3}, \frac{2}{3}\right),\ G_{21} = \left(\frac{1}{9}, \frac{2}{9}\right),\ G_{22} = \left(\frac{7}{9}, \frac{8}{9}\right),\ \text{and so on;}$$

that is, G_{mi} is the ith open interval removed from $[0, 1]$ at the mth step of the process ($i = 1, 2, \ldots, 2^{m-1}$, $m = 1, 2, \ldots$ ad infinitum).

Define $F: [0, 1] \to E^1$ as follows:

(i) $F(0) = 0$;

(ii) if $x \in G_{mi}$, then $F(x) = \dfrac{2i-1}{2^m}$; and

[2] Note that $f(p^-)$ and $f(p^+)$ may not exist if f is not monotone. See Examples (c) and (f) in §1.

(iii) if x is in *none* of the G_{mi} (i.e., $x \in P$), then

$$F(x) = \sup\Big\{F(y) \mid y \in \bigcup_{m,i} G_{mi},\ y < x\Big\}.$$

Show that F is nondecreasing and continuous on $[0, 1]$. (F is called *Cantor's function*.)

7. Restate Theorem 3 for the case where f is monotone on A, where A is a (*not necessarily open*) interval. How about the endpoints of A?

§6. Compact Sets

We now pause to consider a very important kind of sets. In Chapter 3, §16, we showed that every sequence $\{\bar{z}_m\}$ taken from a closed interval $[\bar{a}, \bar{b}]$ in E^n must cluster *in it* (Note 1 of Chapter 3, §16).[1] There are other sets with the same remarkable property. This leads us to the following definition.

Definition 1.

A set $A \subseteq (S, \rho)$ is said to be *sequentially compact* (briefly *compact*) iff every sequence $\{x_m\} \subseteq A$ clusters at some point p *in* A.

If all of S is compact, we say that the *metric space* (S, ρ) is compact.[2]

Examples.

(a) *Each closed interval in* E^n *is compact* (see above).

(a′) However, *nonclosed* intervals, and E^n itself, are not compact.

For example, the sequence $x_n = 1/n$ is in $(0, 1] \subset E^1$, but clusters only at 0, *outside* $(0, 1]$. As another example, the sequence $x_n = n$ has *no* cluster points in E^1. Thus $(0, 1]$ and E^1 fail to be compact (even though E^1 is *complete*); similarly for E^n (*and C^n).

(b) Any *finite* set $A \subseteq (S, \rho)$ is compact. Indeed, an infinite sequence in such a set must have at least one *infinitely repeating* term $p \in A$. Then by definition, this p is a cluster point (see Chapter 3, §14, Note 1).

(c) The *empty set* is "vacuously" compact (it contains *no* sequences).

(d) E^* is compact. See Example (g) in Chapter 3, §14.

Other examples can be derived from the theorems that follow.

[1] Think of $[\bar{a}, \bar{b}]$ as of a container so "compact" that it "squeezes" into clustering any sequence that is inside it, *and it supplies the cluster point*.

[2] Hence A is compact iff (A, ρ) is compact as a subspace of (S, ρ). Note that $\{x_m\}$ clusters at p iff there is a subsequence $x_{m_k} \to p$ (Chapter 3, §16, Theorem 1).

Theorem 1. *If a set $B \subseteq (S, \rho)$ is compact, so is any closed subset $A \subseteq B$.*

Proof. We must show that each sequence $\{x_m\} \subseteq A$ clusters at some $p \in A$. However, as $A \subseteq B$, $\{x_m\}$ is also in B, so by the compactness of B, it clusters at some $p \in B$. Thus it remains to show that $p \in A$ as well.

Now by Theorem 1 of Chapter 3, §16, $\{x_m\}$ has a subsequence $x_{m_k} \to p$. As $\{x_{m_k}\} \subseteq A$ and A is *closed*, this implies $p \in A$ (Theorem 4 in Chapter 3, §16). □

Theorem 2. *Every compact set $A \subseteq (S, \rho)$ is closed.*

Proof. Given that A is compact, we must show (by Theorem 4 in Chapter 3, §16) that A contains the limit of each *convergent* sequence $\{x_m\} \subseteq A$.

Thus let $x_m \to p$, $\{x_m\} \subseteq A$. As A is compact, the sequence $\{x_m\}$ clusters at some $q \in A$, i.e., has a subsequence $x_{m_k} \to q \in A$. However, the limit of the subsequence must be the same as that of the entire sequence. Thus $p = q \in A$; i.e., p is *in* A, as required. □

Theorem 3. *Every compact set $A \subseteq (S, \rho)$ is bounded.*

Proof. By Problem 3 in Chapter 3, §13, it suffices to show that A is contained in some *finite union of globes*. Thus we fix some arbitrary radius $\varepsilon > 0$ and, seeking a contradiction, assume that A cannot be covered by *any finite number* of globes of that radius.

Then if $x_1 \in A$, the globe $G_{x_1}(\varepsilon)$ does not cover A, so there is a point $x_2 \in A$ such that

$$x_2 \notin G_{x_1}(\varepsilon), \text{ i.e., } \rho(x_1, x_2) \geq \varepsilon.$$

By our assumption, A is not even covered by $G_{x_1}(\varepsilon) \cup G_{x_2}(\varepsilon)$. Thus there is a point $x_3 \in A$ with

$$x_3 \notin G_{x_1}(\varepsilon) \text{ and } x_3 \notin G_{x_2}(\varepsilon), \text{ i.e., } \rho(x_3, x_1) \geq \varepsilon \text{ and } \rho(x_3, x_2) \geq \varepsilon.$$

Again, A is not covered by $\bigcup_{i=1}^{3} G_{x_i}(\varepsilon)$, so there is a point $x_4 \in A$ *not* in that union; its distances from x_1, x_2, and x_3 must therefore be $\geq \varepsilon$.

Since A is never covered by any finite number of ε-globes, we can continue this process indefinitely (by induction) and thus select an infinite sequence $\{x_m\} \subseteq A$, with all its terms at least ε-apart from each other.

Now as A is compact, this sequence must have a convergent subsequence $\{x_{m_k}\}$, which is then certainly Cauchy (by Theorem 1 of Chapter 3, §17). This is impossible, however, since its terms are at distances $\geq \varepsilon$ from each other, contrary to Definition 1 in Chapter 3, §17. This contradiction completes the proof. □

Note 1. We have actually proved more than was required, namely, that *no matter how small $\varepsilon > 0$ is, A can be covered by finitely many globes of radius*

ε with centers in A. This property is called *total boundedness* (Chapter 3, §13, Problem 4).

Note 2. Thus all compact sets are closed and bounded. *The converse fails in metric spaces in general* (see Problem 2 below). In E^n (*and C^n), however, the converse is likewise true, as we show next.

Theorem 4. *In E^n (*and C^n) a set is compact iff it is closed and bounded.*

Proof. In fact, if a set $A \subseteq E^n$ ($^*C^n$) is bounded, then by the Bolzano–Weierstrass theorem, each sequence $\{x_m\} \subseteq A$ has a convergent subsequence $x_{m_k} \to p$. If A is also closed, the limit point p must belong to A itself.

Thus each sequence $\{x_m\} \subseteq A$ clusters at some p *in* A, so A is compact.

The converse is obvious. □

Note 3. In particular, *every closed globe in E^n (*or C^n) is compact* since it is bounded and closed (Chapter 3, §12, Example (6)), so Theorem 4 applies.

We conclude with an important theorem, due to G. Cantor.

Theorem 5 (Cantor's principle of nested closed sets)**.** *Every contracting sequence of nonvoid compact sets,*

$$F_1 \supseteq F_2 \supseteq \cdots \supseteq F_m \supseteq \cdots,$$

in a metric space (S, ρ) has a nonvoid intersection; *i.e.*, *some p belongs to* all F_m.

For complete *sets F_m, this holds as well, provided the diameters of the sets F_m tend to* 0: $dF_m \to 0$.

Proof. We prove the theorem for *complete* sets first.

As $F_m \neq \emptyset$, we can pick a point x_m from each F_m to obtain a sequence $\{x_m\}$, $x_m \in F_m$. Since $dF_m \to 0$, it is easy to see that $\{x_m\}$ is a *Cauchy sequence*. (The details are left to the reader.) Moreover,

$$(\forall\, m) \quad x_m \in F_m \subseteq F_1.$$

Thus $\{x_m\}$ is a Cauchy sequence in F_1, a *complete* set (by assumption).

Therefore, by the definition of completeness (Chapter 3, §17), $\{x_m\}$ has a limit $p \in F_1$. This limit remains the same if we drop a finite number of terms, say, the first $m-1$ of them. Then we are left with the sequence $x_m, x_{m+1}, \ldots,$ which, by construction, is *entirely contained in F_m* (why?), with the same limit p. Then, however, the completeness of F_m implies that $p \in F_m$ as well. As m is arbitrary here, it follows that $(\forall\, m)\ p \in F_m$, i.e.,

$$p \in \bigcap_{m=1}^{\infty} F_m, \text{ as claimed.}$$

The proof for *compact* sets is analogous and even simpler. Here $\{x_m\}$ need not be a Cauchy sequence. Instead, using the compactness of F_1, we select from $\{x_m\}$ a subsequence $x_{m_k} \to p \in F_1$ and then proceed as above. □

Note 4. In particular, in E^n we may let the sets F_m be *closed intervals* (since they are compact). Then Theorem 5 yields the *principle of nested intervals*: *Every contracting sequence of* closed *intervals in E^n has a nonempty intersection.* (For an independent proof, see Problem 8 below.)

Problems on Compact Sets

1. Complete the missing details in the proof of Theorem 5.

2. Verify that any infinite set in a *discrete* space is closed and bounded but *not compact.*

[Hint: In such a space *no* sequence of distinct terms clusters.]

3. Show that E^n is not compact, in three ways:

(i) from definitions (as in Example (a$'$));

(ii) from Theorem 4; and

(iii) from Theorem 5, by finding in E^n a contracting sequence of infinite closed sets with a *void* intersection. For example, in E^1 take the closed sets $F_m = [m, +\infty)$, $m = 1, 2, \ldots$. (*Are* they closed?)

4. Show that E^* is compact *under the metric* ρ' defined in Problems 5 and 6 in Chapter 3, §11. Is E^1 a compact set under *that* metric?

[Hint: For the first part, use Theorem 2 of Chapter 2, §13, noting that G_q is also a globe *under* ρ'. For the second, consider the sequence $x_n = n$.]

5. Show that a set $A \subseteq (S, \rho)$ is compact iff every infinite subset $B \subseteq A$ has a cluster point $p \in A$.

[Hint: Select from B a sequence $\{x_m\}$ of *distinct* terms. Then the cluster points of $\{x_m\}$ are also those of B. (Why?)]

6. Prove the following.

(i) If A and B are compact, so is $A \cup B$, and similarly for *unions of n sets.*

(ii) If the sets A_i $(i \in I)$ are compact, so is $\bigcap_{i \in I} A_i$, even if I is *infinite.*

Disprove (i) for *unions* of infinitely many sets by a counterexample.

[Hint: For (ii), verify first that $\bigcap_{i \in I} A_i$ is sequentially *closed.* Then use Theorem 1.]

7. Prove that if $x_m \to p$ in (S, ρ), then the set

$$B = \{p, x_1, x_2, \ldots, x_m, \ldots\}$$

is compact.

[Hint: If B is finite, see Example (b). If not, use Problem 5, noting that any infinite subset of B defines a *subsequence* $x_{m_k} \to p$, so it clusters at p.]

8. Prove, independently, the *principle of nested intervals* in E^n, i.e., Theorem 5 with

$$F_m = [\bar{a}_m, \bar{b}_m] \subseteq E^n,$$

where

$$\bar{a}_m = (a_{m1}, \ldots, a_{mn}) \text{ and } \bar{b}_m = (b_{m1}, \ldots, b_{mn}).$$

[Hint: As $F_{m+1} \subseteq F_m$, $\bar{a}_{m+1}$ and $\bar{b}_{m+1}$ *are in* F_m; hence by properties of closed intervals,

$$a_{mk} \leq a_{m+1,\,k} \leq b_{m+1,\,k} \leq b_{mk}, \quad k = 1, 2, \ldots, n.$$

Fixing k, let A_k be the set of all a_{mk}, $m = 1, 2, \ldots$. Show that A_k is bounded above by each b_{mk}, so let $p_k = \sup A_k$ in E^1. Then

$$(\forall\, m) \quad a_{mk} \leq p_k \leq b_{mk}. \text{ (Why?)}$$

Unfixing k, obtain such inequalities for $k = 1, 2, \ldots, n$. Let $\bar{p} = (p_1, \ldots, p_k)$. Then

$$(\forall\, m) \quad \bar{p} \in [\bar{a}_m, \bar{b}_m], \text{ i.e., } \bar{p} \in \bigcap F_m, \text{ as required.}$$

Note that the theorem fails for nonclosed intervals, even in E^1; e.g., take $F_m = (0, 1/m]$ and show that $\bigcap_m F_m = \emptyset$.]

9. From Problem 8, obtain a new proof of the Bolzano–Weierstrass theorem.

[Hint: Let $\{\bar{x}_m\} \in [\bar{a}, \bar{b}] \subseteq E^n$; put $F_0 = [\bar{a}, \bar{b}]$ and set

$$dF_0 = \rho(\bar{a}, \bar{b}) = d \quad \text{(diagonal of } F_0\text{)}.$$

Bisecting the edges of F_0, subdivide F_0 into 2^n intervals of diagonal $d/2$;[3] one of them must contain infinitely many x_m. (Why?) Let F_1 be one such interval; *make it closed* and subdivide it into 2^n subintervals of diagonal $d/2^2$. One of them, F_2, contains infinitely many x_m; make it closed, etc.

Thus obtain a contracting sequence of closed intervals F_m with

$$dF_m = \frac{d}{2^m}, \quad m = 1, 2, \ldots.$$

From Problem 8, obtain

$$\bar{p} \in \bigcap_{m=1}^{\infty} F_m.$$

Show that $\{\bar{x}_m\}$ clusters at $\bar{p}$.]

⇒**10.** Prove the *Heine–Borel theorem*: *If a closed interval* $F_0 \subset E^n$ *is covered by a family of* open *sets* G_i $(i \in I)$, *i.e.*,

$$F_0 \subseteq \bigcup_{i \in I} G_i,$$

then it can always be covered by a finite number of these G_i.

[Outline of proof: Let $dF_0 = d$. Seeking a contradiction, suppose F_0 *cannot* be covered by any finite number of the G_i.

[3] This is achieved by drawing n planes perpendicular to the axes (Chapter 3, §§4–6).

As in Problem 9, subdivide F_0 into 2^n intervals of diagonal $d/2$. *At least one of them cannot be covered by finitely many* G_i. (Why?) Choose one such interval, *make it closed*, call it F_1, and subdivide it into 2^n subintervals of diagonal $d/2^2$. One of these, F_2, cannot be covered by finitely many G_i; make it closed and repeat the process indefinitely.

Thus obtain a contracting sequence of closed intervals F_m with

$$dF_m = \frac{d}{2^m}, \quad m = 1, 2, \ldots.$$

From Problem 8 (or Theorem 5), get $\bar{p} \in \bigcap F_m$.

As $\bar{p} \in F_0$, $\bar{p}$ is in one of the G_i; call it G. As G is *open*, $\bar{p}$ is its *interior* point, so let $G \supseteq G_{\bar{p}}(\varepsilon)$. Now take m so large that $d/2^m = dF_m < \varepsilon$. Show that then

$$F_m \subseteq G_{\bar{p}}(\varepsilon) \subseteq G.$$

Thus (*contrary to our choice of the* F_m) F_m is covered by a *single* set G_i. This contradiction completes the proof.]

11. Prove that if $\{x_m\} \subseteq A \subseteq (S, \rho)$ and A is compact, then $\{x_m\}$ converges iff it has a *single* cluster point.

[Hint: Proceed as in Problem 12 of Chapter 3, §16.]

12. Prove that if $\emptyset \neq A \subseteq (S, \rho)$ and A is compact, there are two points $p, q \in A$ such that $dA = \rho(p, q)$.

[Hint: As A is bounded (Theorem 3), $dA < +\infty$. By the properties of suprema,

$$(\forall n)\ (\exists x_n, y_n \in A) \quad dA - \frac{1}{n} < \rho(x_n, y_n) \leq dA. \quad \text{(Explain!)}$$

By compactness, $\{x_n\}$ has a subsequence $x_{n_k} \to p \in A$. For brevity, put $x'_k = x_{n_k}$, $y'_k = y_{n_k}$. Again, $\{y'_k\}$ has a subsequence $y'_{k_m} \to q \in A$. Also,

$$dA - \frac{1}{n_{k_m}} < \rho(x'_{k_m}, y'_{k_m}) \leq dA.$$

Passing to the limit (as $m \to +\infty$), obtain

$$dA \leq \rho(p, q) \leq dA$$

by Theorem 4 in Chapter 3, §15.]

13. Given nonvoid sets $A, B \subseteq (S, \rho)$, define

$$\rho(A, B) = \inf\{\rho(x, y) \mid x \in A,\ y \in B\}.$$

Prove that if A and B are compact and nonempty, there are $p \in A$ and $q \in B$ such that $\rho(p, q) = \rho(A, B)$. Give an example to show that this may fail if A and B are not compact (even if they are closed in E^1).

[Hint: For the first part, proceed as in Problem 12.]

14. Prove that every compact set is complete. Disprove the converse by examples.

*§7. More on Compactness

Another useful approach to compactness is based on the notion of a *covering* of a set (already encountered in Problem 10 in §6). We say that a set F is *covered* by a family of sets G_i $(i \in I)$ iff

$$F \subseteq \bigcup_{i\in I} G_i.$$

If this is the case, $\{G_i\}$ is called a *covering* of F. If the sets G_i are *open*, we call the set family $\{G_i\}$ an *open covering*. The covering $\{G_i\}$ is said to be finite (infinite, countable, etc.) iff the number of the sets G_i is.

If $\{G_i\}$ is an open covering of F, then each point $x \in F$ is in some G_i and is its *interior* point (for G_i is *open*), so there is a globe $G_x(\varepsilon_x) \subseteq G_i$. In general, the radii ε_x of these globes depend on x, i.e., are different for different points $x \in F$. If, however, they can be chosen *all equal to some* ε, then this ε is called a *Lebesgue number* for the covering $\{G_i\}$ (so named after Henri Lebesgue). Thus ε is a Lebesgue number iff *for every* $x \in F$*, the globe* $G_x(\varepsilon)$ *is contained in some* G_i. We now obtain the following theorem.

Theorem 1 (Lebesgue)**.** *Every open covering* $\{G_j\}$ *of a sequentially compact set* $F \subseteq (S, \rho)$ *has at least one Lebesgue number* ε*. In symbols*,

$$(\exists\, \varepsilon > 0)\ (\forall\, x \in F)\ (\exists\, i) \quad G_x(\varepsilon) \subseteq G_i. \tag{1}$$

Proof. Seeking a contradiction, assume that (1) *fails*, i.e., its *negation* holds. As was explained in Chapter 1, §§1–3, this negation is

$$(\forall\, \varepsilon > 0)\ (\exists\, x_\varepsilon \in F)\ (\forall\, i) \quad G_{x_\varepsilon}(\varepsilon) \nsubseteq G_i$$

(where we write x_ε for x since here x *may depend on* ε). As this is supposed to hold for *all* $\varepsilon > 0$, we take successively

$$\varepsilon = 1,\ \frac{1}{2},\ \ldots,\ \frac{1}{n},\ \ldots.$$

Then, replacing "x_ε" by "x_n" for convenience, we obtain

$$(\forall\, n)\ (\exists\, x_n \in F)\ (\forall\, i) \quad G_{x_n}\left(\frac{1}{n}\right) \nsubseteq G_i. \tag{2}$$

Thus for each n, there is some $x_n \in F$ such that the globe $G_{x_n}(\frac{1}{n})$ is *not* contained in *any* G_i. We fix such an $x_n \in F$ for each n, thus obtaining a sequence $\{x_n\} \subseteq F$. As F is compact (by assumption), this sequence clusters at some $p \in F$.

The point p, being in F, must be in some G_i (call it G), *together with some globe* $G_p(r) \subseteq G$. As p is a cluster point, even the smaller globe $G_p(\frac{r}{2})$ contains *infinitely many* x_n. Thus we may choose n so large that $\frac{1}{n} < \frac{r}{2}$ and $x_n \in G_p(\frac{r}{2})$. For that n, $G_{x_n}(\frac{1}{n}) \subseteq G_p(r)$ because

$$\left(\forall\, x \in G_{x_n}\left(\frac{1}{n}\right)\right) \quad \rho(x,p) \le \rho(x,x_n) + \rho(x_n,p) < \frac{1}{n} + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r.$$

As $G_p(r) \subseteq G$ (by construction), we certainly have

$$G_{x_n}\left(\frac{1}{n}\right) \subseteq G_p(r) \subseteq G.$$

However, this is impossible since by (2) *no* $G_{x_n}(\frac{1}{n})$ is contained in *any* G_i. This contradiction completes the proof. $\square$

Our next theorem might serve as an alternative definition of compactness. In fact, in *topology* (which studies spaces more general than metric spaces), this *is* is the basic definition of compactness. It generalizes Problem 10 in §6.

Theorem 2 (generalized Heine–Borel theorem)**.** *A set $F \subseteq (S,\, \rho)$ is compact iff every open covering of F has a finite subcovering.*

That is, whenever F is covered by a family of open sets G_i $(i \in I)$, F can also be covered by a finite *number of these G_i.*

Proof. Let F be sequentially compact, and let $F \subseteq \bigcup G_i$, all G_i open. We have to show that $\{G_i\}$ reduces to a finite subcovering.

By Theorem 1, $\{G_i\}$ has a Lebesgue number ε satisfying (1). We fix this $\varepsilon > 0$. Now by Note 1 in §6, we can cover F by a finite number of ε-globes,

$$F \subseteq \bigcup_{k=1}^{n} G_{x_k}(\varepsilon), \quad x_k \in F.$$

Also by (1), each $G_{x_k}(\varepsilon)$ is contained in some G_i; call it G_{i_k}. With the G_{i_k} so fixed, we have

$$F \subseteq \bigcup_{k=1}^{n} G_{x_k}(\varepsilon) \subseteq \bigcup_{k=1}^{n} G_{i_k}.$$

Thus the sets G_{i_k} constitute the desired finite subcovering, and the "only if" in the theorem is proved.

Conversely, assume the condition stated in the theorem. We have to show that F is sequentially compact, i.e., that every sequence $\{x_m\} \subseteq F$ clusters at some $p \in F$.

Seeking a contradiction, suppose F contains *no* cluster points of $\{x_m\}$. Then by definition, each point $x \in F$ is in some globe G_x *containing at most finitely many* x_m. The set F is covered by these open globes, hence also by finitely many of them (by our assumption). Then, however, F contains *at most finitely many* x_m (namely, those contained in the so-selected globes), whereas the sequence $\{x_m\} \subseteq F$ was assumed *infinite*. This contradiction completes the proof. $\square$

§8. Continuity on Compact Sets. Uniform Continuity

I. Some additional important theorems apply to functions that are continuous on a *compact* set (see §6).

Theorem 1. *If a function* $f\colon A \to (T, \rho')$, $A \subseteq (S, \rho)$, *is relatively continuous on a compact set* $B \subseteq A$, *then* $f[B]$ *is a compact set in* (T, ρ'). *Briefly,*

the continuous image of a compact set is compact.

Proof. To show that $f[B]$ is compact, we take any sequence $\{y_m\} \subseteq f[B]$ and prove that it clusters at some $q \in f[B]$.

As $y_m \in f[B]$, $y_m = f(x_m)$ for some x_m in B. We pick such an $x_m \in B$ for each y_m, thus obtaining a sequence $\{x_m\} \subseteq B$ with

$$f(x_m) = y_m, \quad m = 1, 2, \ldots.$$

Now by the assumed compactness of B, the sequence $\{x_m\}$ must cluster at some $p \in B$. Thus it has a subsequence $x_{m_k} \to p$. As $p \in B$, the function f is relatively continuous at p over B (by assumption). Hence by the sequential criterion (§2), $x_{m_k} \to p$ implies $f(x_{m_k}) \to f(p)$; i.e.,

$$y_{m_k} \to f(p) \in f[B].$$

Thus $q = f(p)$ is the desired cluster point of $\{y_m\}$. $\square$

This theorem can be used to prove the compactness of various sets.

Examples.

(1) A *closed line segment* $L[\bar{a}, \bar{b}]$ in E^n (*and in other normed spaces) is compact, for, by definition,

$$L[\bar{a}, \bar{b}] = \{\bar{a} + t\vec{u} \mid 0 \leq t \leq 1\}, \text{ where } \vec{u} = \bar{b} - \bar{a}.$$

Thus $L[\bar{a}, \bar{b}]$ is *the image of the compact interval* $[0, 1] \subseteq E^1$ under the map $f\colon E^1 \to E^n$, given by $f(t) = \bar{a} + t\vec{u}$, which is continuous by Theorem 3 of §3. (Why?)

(2) The closed solid ellipsoid in E^3,

$$\left\{(x, y, z) \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\},$$

is compact, being the image of a compact globe under a suitable continuous map. The details are left to the reader as an exercise.

Lemma 1. *Every nonvoid compact set $F \subseteq E^1$ has a maximum and a minimum.*

Proof. By Theorems 2 and 3 of §6, F is closed and bounded. Thus F has an infimum and a supremum in E^1 (by the completeness axiom), say, $p = \inf F$ and $q = \sup F$. It remains to show that $p, q \in F$.

Assume the opposite, say, $q \notin F$. Then by properties of suprema, each globe $G_q(\delta) = (q - \delta, q + \delta)$ contains some $x \in B$ (specifically, $q - \delta < x < q$) *other than* q (for $q \notin B$, while $x \in B$). Thus

$$(\forall \delta > 0) \quad F \cap G_{\neg q}(\delta) \neq \emptyset;$$

i.e., F clusters at q and hence *must contain* q (being *closed*). However, since $q \notin F$, this is the desired contradiction, and the lemma is proved. □

The next theorem has many important applications in analysis.

Theorem 2 (Weierstrass)**.**

(i) *If a function $f\colon A \to (T, \rho')$ is relatively continuous on a compact set $B \subseteq A$, then f is bounded on B; i.e., $f[B]$ is bounded.*

(ii) *If, in addition, $B \neq \emptyset$ and f is* real *($f\colon A \to E^1$), then $f[B]$ has a maximum and a minimum; i.e., f attains a largest and a least value at some points of B.*

Proof. Indeed, by Theorem 1, $f[B]$ is compact, so it is bounded, as claimed in (i).

If further $B \neq \emptyset$ and f is real, then $f[B]$ is a nonvoid compact set in E^1, so by Lemma 1, it has a maximum and a minimum in E^1. Thus all is proved. □

Note 1. This and the other theorems of this section hold, in particular, if B is a *closed interval in* E^n or a *closed globe in* E^n (**or* C^n) (because these sets are compact—see the examples in §6). This may fail, however, if B is not compact, e.g., if $B = (\bar{a}, \bar{b})$. For a counterexample, see Problem 11 in Chapter 3, §13.

Theorem 3. *If a function $f\colon A \to (T, \rho')$, $A \subseteq (S, \rho)$, is relatively continuous on a compact set $B \subseteq A$ and is one to one on B (i.e., when restricted to B), then its inverse, f^{-1}, is continuous on $f[B]$.*[1]

Proof. To show that f^{-1} is continuous at each point $q \in f[B]$, we apply the sequential criterion (Theorem 1 in §2). Thus we fix a sequence $\{y_m\} \subseteq f[B]$, $y_m \to q \in f[B]$, and prove that $f^{-1}(y_m) \to f^{-1}(q)$.

[1] Note that f need not be one to one on *all* of its domain A, only on B. Thus f^{-1} need not be a mapping on $f[A]$, but it is one on $f[B]$. (We use "f^{-1}" here to denote the inverse of f so restricted.)

Let $f^{-1}(y_m) = x_m$ and $f^{-1}(q) = p$ so that

$$y_m = f(x_m),\ q = f(p), \text{ and } x_m,\ p \in B.$$

We have to show that $x_m \to p$, i.e., that

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, m > k) \quad \rho(x_m,\, p) < \varepsilon.$$

Seeking a contradiction, suppose this *fails*, i.e., its negation holds. Then (see Chapter 1, §§1–3) there is an $\varepsilon > 0$ such that

$$(\forall\, k)\ (\exists\, m_k > k) \quad \rho(x_{m_k}, p) \geq \varepsilon, \tag{1}$$

where we write "m_k" for "m" to stress that the m_k may be different for different k. Thus by (1), we fix some m_k for each k so that (1) holds, *choosing step by step*,

$$m_{k+1} > m_k, \quad k = 1, 2, \ldots .$$

Then the x_{m_k} form a subsequence of $\{x_m\}$, and the corresponding $y_{m_k} = f(x_{m_k})$ form a subsequence of $\{y_m\}$. Henceforth, for brevity, let $\{x_m\}$ and $\{y_m\}$ themselves denote these two subsequences. Then as before, $x_m \in B$, $y_m = f(x_m) \in f[B]$, and $y_m \to q$, $q = f(p)$. Also, by (1),

$$(\forall\, m) \quad \rho(x_m, p) \geq \varepsilon \ (x_m \text{ stands for } x_{m_k}). \tag{2}$$

Now as $\{x_m\} \subseteq B$ and B is compact, $\{x_m\}$ has a (sub)subsequence

$$x_{m_i} \to p' \text{ for some } p' \in B.$$

As f is relatively continuous on B, this implies

$$f(x_{m_i}) = y_{m_i} \to f(p').$$

However, the subsequence $\{y_{m_i}\}$ must have the same limit as $\{y_m\}$, i.e., $f(p)$. Thus $f(p') = f(p)$, whence $p = p'$ (for f is one to one on B), so $x_{m_i} \to p' = p$.

This contradicts (2), however, and thus the proof is complete.[2] □

Examples (continued)**.**

(3) For a fixed $n \in N$, define $f\colon [0, +\infty) \to E^1$ by

$$f(x) = x^n.$$

Then f is one to one (strictly increasing) and continuous (being a *monomial*; see §3). Thus by Theorem 3, f^{-1} (the nth root function) is relatively continuous on each interval

$$f[[a, b]] = [a^n, b^n],$$

hence on $[0, +\infty)$.

[2] We call f *bicontinuous* if (as in our case) both f and f^{-1} are continuous.

See also Example (a) in §6 and Problem 1 below.

II. Uniform Continuity. If f is relatively continuous on B, then by definition,

$$(\forall\,\varepsilon > 0)\ (\forall\,p \in B)\ (\exists\,\delta > 0)\ (\forall\,x \in B \cap G_p(\delta))\quad \rho'(f(x),\, f(p)) < \varepsilon. \tag{3}$$

Here, in general, δ depends on *both* ε and p (see Problem 4 in §1); that is, given $\varepsilon > 0$, some values of δ may fit a given p but fail (3) for other points.

It may occur, however, that *one and the same* δ (depending on ε only) satisfies (3) for *all* $p \in B$ simultaneously, so that we have the stronger formula

$$(\forall\,\varepsilon > 0)\ (\exists\,\delta > 0)\ (\forall\,p,\,x \in B \mid \rho(x,\,p) < \delta)\quad \rho'(f(x),\, f(p)) < \varepsilon.^{3} \tag{4}$$

Definition 1.

If (4) is true, we say that f is *uniformly continuous* on B.

Clearly, this *implies* (3), but the converse fails.[4]

Theorem 4. *If a function* $f\colon A \to (T,\,\rho')$, $A \subseteq (S,\,\rho)$, *is relatively continuous on a compact set* $B \subset A$, *then* f *is also uniformly continuous on* B.

Proof (by contradiction)**.** Suppose f is relatively continuous on B, but (4) *fails*. Then there is an $\varepsilon > 0$ such that

$$(\forall\,\delta > 0)\ (\exists\,p,\,x \in B)\quad \rho(x,\,p) < \delta, \text{ and yet } \rho'(f(x),\, f(p)) \ge \varepsilon;$$

here p and x depend on δ. We fix such an ε and let

$$\delta = 1,\, \frac{1}{2},\, \ldots,\, \frac{1}{m},\, \ldots.$$

Then for each δ (i.e., each m), we get two points $x_m,\, p_m \in B$ with

$$\rho(x_m,\, p_m) < \frac{1}{m} \tag{5}$$

and

$$\rho'(f(x_m),\, f(p_m)) \ge \varepsilon, \quad m = 1,\, 2,\, \ldots. \tag{6}$$

Thus we obtain two sequences, $\{x_m\}$ and $\{p_m\}$, in B. As B is compact, $\{x_m\}$ has a subsequence $x_{m_k} \to q$ $(q \in B)$. For simplicity, let it be $\{x_m\}$ itself; thus

$$x_m \to q, \quad q \in B.$$

[3] In other words, $f(x)$ and $f(p)$ are ε-close for *any* $p,\, x \in B$ with $\rho(p,\, x) < \delta$.
[4] See Example (h) below.

Hence by (5), it easily follows that also $p_m \to q$ (because $\rho(x_m, p_m) \to 0$; see Problem 4 in Chapter 3, §17). By the assumed relative continuity of f on B, it follows that

$$f(x_m) \to f(q) \text{ and } f(p_m) \to f(q) \text{ in } (T, \rho').$$

This, in turn, implies that $\rho'(f(x_m), f(p_m)) \to 0$, *which is impossible*, in view of (6). This contradiction completes the proof. □

One type of uniformly continuous functions are so-called *contraction mappings*. We define them in Example (a) below and hence derive a few noteworthy special cases. Some of them are so-called *isometries* (see Problems, footnote 5).

Examples.

(a) A function $f\colon A \to (T, \rho')$, $A \subseteq (S, \rho)$, is called a *contraction map* (on A) iff

$$\rho(x, y) \geq \rho'(f(x), f(y)) \text{ for all } x, y \in A.$$

Any such map is uniformly continuous on A. In fact, given $\varepsilon > 0$, we simply take $\delta = \varepsilon$. Then $(\forall\, x, p \in A)$

$$\rho(x, p) < \delta \text{ implies } \rho'(f(x), f(p)) \leq \rho(x, p) < \delta = \varepsilon,$$

as required in (3).

(b) As a special case, consider the *absolute value map* (*norm map*) given by

$$f(\bar{x}) = |\bar{x}| \text{ on } E^n \text{ (}^*\text{or another normed space).}$$

It is uniformly continuous on E^n because

$$\big||\bar{x}| - |\bar{p}|\big| \leq |\bar{x} - \bar{p}|, \text{ i.e., } \rho'(f(\bar{x}), f(\bar{p})) \leq \rho(\bar{x}, \bar{p}),$$

which shows that f is a *contraction map*, so Example (a) applies.

(c) Other examples of contraction maps are

(1) *constant maps* (see §1, Example (a)) and

(2) *projection maps* (see the proof of Theorem 3 in §3).

Verify!

(d) Define $f\colon E^1 \to E^1$ by

$$f(x) = \sin x$$

By elementary trigonometry, $|\sin x| \leq |x|$. Thus $(\forall\, x,\, p \in E^1)$

$$\begin{aligned} |f(x) - f(p)| &= |\sin x - \sin p| \\ &= 2\left|\sin \frac{1}{2}(x-p) \cdot \cos \frac{1}{2}(x+p)\right| \\ &\leq 2\left|\sin \frac{1}{2}(x-p)\right| \\ &\leq 2 \cdot \frac{1}{2}\,|x-p| = |x-p|, \end{aligned}$$

and f is a contraction map again. Hence *the sine function is uniformly continuous on* E^1; similarly for the cosine function.

(e) Given $\emptyset \neq A \subseteq (S,\, \rho)$, define $f\colon S \to E^1$ by

$$f(x) = \rho(x,\, A) \text{ where } \rho(x,\, A) = \inf_{y \in A} \rho(x,\, y).$$

It is easy to show that

$$(\forall\, x,\, p \in S) \quad \rho(x,\, A) \leq \rho(x,\, p) + \rho(p,\, A),$$

i.e.,

$$f(x) \leq \rho(p,\, x) + f(p), \text{ or } f(x) - f(p) \leq \rho(p,\, x).$$

Similarly, $f(p) - f(x) \leq \rho(p,\, x)$. Thus

$$|f(x) - f(p)| \leq \rho(p,\, x);$$

i.e., f is uniformly continuous (being a contraction map).

(f) The *identity map* $f\colon (S,\, \rho) \to (S,\, \rho)$, given by

$$f(x) = x,$$

is uniformly continuous on S since

$$\rho(f(x),\, f(p)) = \rho(x,\, p) \text{ (a contraction map!).}$$

However, even relative continuity could fail if the metric in the *domain space* S were not the same as in S when regarded as the *range space* (e.g., make ρ' discrete!)

(g) Define $f\colon E^1 \to E^1$ by

$$f(x) = a + bx \quad (b \neq 0).$$

Then

$$(\forall\, x,\, p \in E^1) \quad |f(x) - f(p)| = |b|\,|x-p|;$$

i.e.,

$$\rho(f(x),\, f(p)) = |b|\,\rho(x,\, p).$$

Thus, given $\varepsilon > 0$, take $\delta = \varepsilon/|b|$. Then

$$\rho(x, p) < \delta \Longrightarrow \rho(f(x), f(p)) = |b|\, \rho(x, p) < |b|\, \delta = \varepsilon,$$

proving uniform continuity.

(h) Let

$$f(x) = \frac{1}{x} \quad \text{on } B = (0, +\infty).$$

Then f is continuous on B, *but not uniformly so*. Indeed, we can prove the *negation* of (4), i.e.,

$$(\exists\, \varepsilon > 0)\ (\forall\, \delta > 0)\ (\exists\, x, p \in B) \quad \rho(x, p) < \delta \text{ and } \rho'(f(x), f(p)) \geq \varepsilon. \tag{4′}$$

Take $\varepsilon = 1$ and *any* $\delta > 0$. We look for x, p such that

$$|x - p| < \delta \text{ and } |f(x) - f(p)| \geq \varepsilon,$$

i.e.,

$$\left|\frac{1}{x} - \frac{1}{p}\right| \geq 1.$$

This is achieved by taking

$$p = \min\left(\delta, \frac{1}{2}\right),\ x = \frac{p}{2}. \quad \text{(Verify!)}$$

Thus (4) fails on $B = (0, +\infty)$, yet *it holds on* $[a, +\infty)$ *for any* $a > 0$. (Verify!)

Problems on Uniform Continuity; Continuity on Compact Sets

1. Prove that if f is relatively continuous on each compact subset of D, then it is relatively continuous on D.
[Hint: Use Theorem 1 of §2 and Problem 7 in §6.]

2. Do Problem 4 in Chapter 3, §17, and thus complete the last details in the proof of Theorem 4.

3. Give an example of a continuous one-to-one map f such that f^{-1} is *not* continuous.
[Hint: Show that *any* map is continuous on a *discrete* space (S, ρ).]

4. Give an example of a continuous function f and a compact set $D \subseteq (T, \rho')$ such that $f^{-1}[D]$ is *not* compact.
[Hint: Let f be *constant* on E^1.]

5. Complete the missing details in Examples (1) and (2) and (c)–(h).

6. Show that every polynomial of degree one on E^n (*or C^n) is uniformly continuous.

7. Show that the arcsine function is uniformly continuous on $[-1, 1]$.
[Hint: Use Example (d) and Theorems 3 and 4.]

⇒8. Prove that if f is uniformly continuous on B, and if $\{x_m\} \subseteq B$ is a *Cauchy sequence*, so is $\{f(x_m)\}$. (Briefly, f *preserves* Cauchy sequences.) Show that this may fail if f is only continuous *in the ordinary sense*. (See Example (h).)

9. Prove that if $f: S \to T$ is uniformly continuous on $B \subseteq S$, and $g: T \to U$ is uniformly continuous on $f[B]$, then the composite function $g \circ f$ is uniformly continuous on B.

10. Show that the functions f and f^{-1} in Problem 5 of Chapter 3, §11 are *contraction maps*,[5] hence uniformly continuous. By Theorem 1, find again that (E^*, ρ') is compact.

11. Let A' be the set of all cluster points of $A \subseteq (S, \rho)$. Let $f: A \to (T, \rho')$ be uniformly continuous on A, and let (T, ρ') be complete.

(i) Prove that $\lim_{x\to p} f(x)$ exists at each $p \in A'$.

(ii) Thus *define* $f(p) = \lim_{x\to p} f(x)$ for each $p \in A' - A$, and show that f so extended is uniformly continuous on the set $\overline{A} = A \cup A'$.[6]

(iii) Consider, in particular, the case $A = (a, b) \subseteq E^1$, so that

$$\overline{A} = A' = [a, b].$$

[Hint: Take any sequence $\{x_m\} \subseteq A$, $x_m \to p \in A'$. As it is Cauchy (why?), so is $\{f(x_m)\}$ by Problem 8. Use Corollary 1 in §2 to prove existence of $\lim_{x\to p} f(x)$. For uniform continuity, use definitions; in case (iii), use Theorem 4.]

12. Prove that if two functions f, g with values in a normed vector space are uniformly continuous on a set B, so also are $f \pm g$ and af for a fixed scalar a.

For *real* functions, prove this also for $f \vee g$ and $f \wedge g$ defined by

$$(f \vee g)(x) = \max(f(x), g(x))$$

and

$$(f \wedge g)(x) = \min(f(x), g(x)).$$

[Hint: After proving the first statements, verify that

$$\max(a, b) = \frac{1}{2}(a + b + |b - a|) \text{ and } \min(a, b) = \frac{1}{2}(a + b - |b - a|)$$

and use Problem 9 and Example (b).]

[5] They even are so-called *isometries*; a map $f: (S, \rho) \to (T, \rho')$ is an *isometry* iff for all x and y in S, $\rho(x, y) = \rho'(f(x), f(y))$.

[6] It is an easier problem to prove *ordinary* continuity. Do that first.

13. Let f be vector valued and h scalar valued, with both uniformly continuous on $B \subseteq (S, \rho)$.

Prove that

(i) if f and h are bounded on B, then hf is uniformly continuous on B;

(ii) the function f/h is uniformly continuous on B if f is bounded on B and h is "*bounded away*" *from* 0 *on* B, i.e.,

$$(\exists\, \delta > 0)\ (\forall\, x \in B) \quad |h(x)| \geq \delta.$$

Give examples to show that without these additional conditions, hf and f/h may not be uniformly continuous (see Problem 14 below).

14. In the following cases, show that f is uniformly continuous on $B \subseteq E^1$, but only continuous (in the ordinary sense) on D, as indicated, with $0 < a < b < +\infty$.

(a) $f(x) = \dfrac{1}{x^2}$; $B = [a, +\infty)$; $D = (0, 1)$.

(b) $f(x) = x^2$; $B = [a, b]$; $D = [a, +\infty)$.

(c) $f(x) = \sin \dfrac{1}{x}$; B and D as in (a).

(d) $f(x) = x \cos x$; B and D as in (b).

15. Prove that if f is uniformly continuous on B, it is so on each subset $A \subseteq B$.

16. For nonvoid sets $A, B \subseteq (S, \rho)$, define

$$\rho(A, B) = \inf\{\rho(x, y) \mid x \in A,\ y \in B\}.$$

Prove that if $\rho(A, B) > 0$ and if f is uniformly continuous on each of A and B, it is so on $A \cup B$.

Show by an example that this fails if $\rho(A, B) = 0$, *even if* $A \cap B = \emptyset$ (e.g., take $A = [0, 1]$, $B = (1, 2]$ in E^1, making f constant on each of A and B).

Note, however, that if A and B are *compact*, $A \cap B = \emptyset$ *implies* $\rho(A, B) > 0$. (Prove it using Problem 13 in §6.) Thus $A \cap B = \emptyset$ suffices in this case.

17. Prove that if f is relatively continuous on each of the disjoint closed sets

$$F_1, F_2, \ldots, F_n,$$

it is relatively continuous on their union

$$F = \bigcup_{k=1}^{n} F_k;$$

hence (see Problem 6 of §6) it is *uniformly* continuous on F if the F_k are compact.

[Hint: Fix any $p \in F$. Then p is in some F_k, say, $p \in F_1$. As the F_k are disjoint, $p \notin F_2, \ldots, F_p$; hence p also is *no cluster point* of any of $F_2, \ldots, F_n$ (for they are *closed*).

Deduce that there is a globe $G_p(\delta)$ disjoint from *each* of $F_2, \ldots, F_n$, so that $F \cap G_p(\delta) = F_1 \cap G_p(\delta)$. From this it is easy to show that relative continuity of f *on* F follows from relative continuity *on* F_1.]

⇒**18.** Let $\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_m$ be fixed points in E^n (*or in another normed space). Let

$$f(t) = \bar{p}_k + (t - k)(\bar{p}_{k+1} - \bar{p}_k)$$

whenever $k \leq t \leq k+1$, $t \in E^1$, $k = 0, 1, \ldots, m-1$.

Show that this defines a uniformly continuous mapping f of the interval $[0, m] \subseteq E^1$ onto the "polygon"

$$\bigcup_{k=0}^{m-1} L[p_k, p_{k+1}].$$

In what case is f one to one? Is f^{-1} uniformly continuous on each $L[p_k, p_{k+1}]$? On the entire polygon?

[Hint: First prove *ordinary* continuity on $[0, m]$ using Theorem 1 of §3. (For the points $1, 2, \ldots, m-1$, consider left and right limits.) Then use Theorems 1–4.]

19. Prove the *sequential criterion for uniform continuity*: A function $f\colon A \to T$ is uniformly continuous on a set $B \subseteq A$ iff for any two (not necessarily convergent) sequences $\{x_m\}$ and $\{y_m\}$ in B, with $\rho(x_m, y_m) \to 0$, we have $\rho'(f(x_m), f(y_m)) \to 0$ (i.e., f preserves *concurrent* pairs of sequences; see Problem 4 in Chapter 3, §17).

§9. The Intermediate Value Property

Definition 1.

A function $f\colon A \to E^*$ is said to have the *intermediate value property*, or *Darboux property*,[1] on a set $B \subseteq A$ iff, *together with any two function values* $f(p)$ *and* $f(p_1)$ $(p, p_1 \in B)$, *it also takes all intermediate values between* $f(p)$ *and* $f(p_1)$ *at some points of* B.

In other words, the image set $f[B]$ contains the entire interval between $f(p)$ and $f(p_1)$ in E^*.

[1] This property is named after Jean Gaston Darboux, who investigated it for *derivatives* (see Chapter 5, §2, Theorem 4).

Note 1. It follows that $f[B]$ itself is a finite or infinite interval in E^*, with endpoints $\inf f[B]$ and $\sup f[B]$. (Verify!)

Geometrically, if $A \subseteq E^1$, this means that the curve $y = f(x)$ meets all horizontal lines $y = q$, for q between $f(p)$ and $f(p_1)$. For example, in Figure 13 in §1, we have a "smooth" curve that cuts each horizontal line $y = q$ between $f(0)$ and $f(p_1)$; so f has the Darboux property on $[0, p_1]$. In Figures 14 and 15, there is a "gap" at p; the property fails. In Example (f) of §1, the property holds on all of E^1 *despite* a discontinuity at 0. Thus it does not *imply* continuity.

Intuitively, it seems plausible that a "continuous curve" must cut all intermediate horizontals. A *precise* proof for functions continuous on an interval, was given independently by Bolzano and Weierstrass (the same as in Theorem 2 of Chapter 3, §16). Below we give a more general version of Bolzano's proof based on the notion of a *convex set* and related concepts.

Definition 2.

A set B in E^n (*or in another normed space) is said to be *convex* iff for each $\bar{a}, \bar{b} \in B$ the line segment $L[\bar{a}, \bar{b}]$ is a subset of B.

A *polygon joining* $\bar{a}$ *and* $\bar{b}$ is any finite union of line segments (a "broken line") of the form

$$\bigcup_{i=0}^{m-1} L[\bar{p}_i, \bar{p}_{i+1}] \text{ with } \bar{p}_0 = \bar{a} \text{ and } \bar{p}_m = \bar{b}.$$

The set B is said to be *polygon connected* (or *piecewise convex*) iff any two points $\bar{a}, \bar{b} \in B$ can be joined by a polygon contained in B.

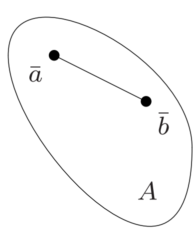

FIGURE 19

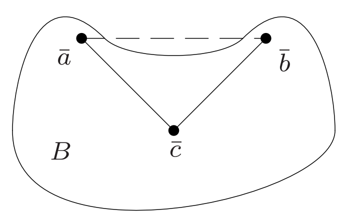

FIGURE 20

Example.

Any globe in E^n (*or in another normed space) is convex, so also is any interval in E^n or in E^*. Figures 19 and 20 represent a convex set A and a polygon-connected set B in E^2 (B is not convex; it has a "cavity").

We shall need a simple lemma that is noteworthy in its own right as well.

Lemma 1 (principle of nested line segments)**.** *Every contracting sequence of closed line segments* $L[\bar{p}_m, \bar{q}_m]$ *in* E^n (**or in any other normed space*) *has a nonvoid intersection*; *i.e.*, *there is a point*

$$\bar{p} \in \bigcap_{m=1}^{\infty} L[\bar{p}_m, \bar{q}_m].$$

Proof. Use Cantor's theorem (Theorem 5 of §6) and Example (1) in §8. □

We are now ready for Bolzano's theorem. The proof to be used is typical of so-called "*bisection proofs*." (See also §6, Problems 9 and 10 for such proofs.)

Theorem 1. *If* $f: B \to E^1$ *is relatively continuous on a polygon-connected set* B *in* E^n (**or in another normed space*), *then* f *has the Darboux property on* B.

In particular, *if* B *is convex and if* $f(\bar{p}) < c < f(\bar{q})$ *for some* $\bar{p}, \bar{q} \in B$, *then there is a point* $\bar{r} \in L(\bar{p}, \bar{q})$ *such that* $f(\bar{r}) = c$.

Proof. First, let B be *convex*. Seeking a contradiction, suppose $\bar{p}, \bar{q} \in B$ with

$$f(\bar{p}) < c < f(\bar{q}),$$

yet $f(\bar{x}) \neq c$ for *all* $\bar{x} \in L(\bar{p}, \bar{q})$.

Let P be the set of all those $\bar{x} \in L[\bar{p}, \bar{q}]$ for which $f(\bar{x}) < c$, i.e.,

$$P = \{\bar{x} \in L[\bar{p}, \bar{q}] \mid f(\bar{x}) < c\},$$

and let

$$Q = \{\bar{x} \in L[\bar{p}, \bar{q}] \mid f(\bar{x}) > c\}.$$

Then $\bar{p} \in P$, $\bar{q} \in Q$, $P \cap Q = \emptyset$, and $P \cup Q = L[\bar{p}, \bar{q}] \subseteq B$. (Why?)

Now let

$$\bar{r}_0 = \frac{1}{2}(\bar{p} + \bar{q})$$

be the midpoint on $L[\bar{p}, \bar{q}]$. Clearly, $\bar{r}_0$ is either in P or in Q. Thus it bisects $L[\bar{p}, \bar{q}]$ into two subsegments, *one of which must have its left endpoint in* P *and its right endpoint in* Q.[2]

We denote this particular closed segment by $L[\bar{p}_1, \bar{q}_1]$, $\bar{p}_1 \in P$, $\bar{q}_1 \in Q$. We then have

$$L[\bar{p}_1, \bar{q}_1] \subseteq L[\bar{p}, \bar{q}] \text{ and } |p_1 - q_1| = \frac{1}{2}|\bar{p} - \bar{q}|. \text{ (Verify!)}$$

Now we bisect $L[\bar{p}_1, \bar{q}_1]$ and repeat the process. Thus let

$$\bar{r}_1 = \frac{1}{2}(\bar{p}_1 + \bar{q}_1).$$

[2] Indeed, if $\bar{r}_0 \in P$, this holds for $L[\bar{r}_0, \bar{q}]$. If $\bar{r}_0 \in Q$, take $L[\bar{p}, \bar{r}_0]$.

By the same argument, we obtain a closed subsegment $L[\bar{p}_2, \bar{q}_2] \subseteq L[\bar{p}_1, \bar{q}_1]$, with $\bar{p}_2 \in P$, $\bar{q}_2 \in Q$, and

$$|\bar{p}_2 - \bar{q}_2| = \frac{1}{2}|\bar{p}_1 - \bar{q}_1| = \frac{1}{4}|\bar{p} - \bar{q}|.$$

Next, we bisect $L[\bar{p}_2, \bar{q}_2]$, and so on. Continuing this process indefinitely, we obtain an infinite contracting sequence of closed line segments $L[\bar{p}_m, \bar{q}_m]$ such that

$$(\forall\, m) \quad \bar{p}_m \in P,\ \bar{q}_m \in Q,$$

and

$$|\bar{p}_m - \bar{q}_m| = \frac{1}{2^m}\,|\bar{p} - \bar{q}| \to 0 \text{ as } m \to +\infty.$$

By Lemma 1, there is a point

$$\bar{r} \in \bigcap_{m=1}^{\infty} L[\bar{p}_m, \bar{q}_m].$$

This implies that

$$(\forall\, m) \quad |\bar{r} - \bar{p}_m| \leq |\bar{p}_m - \bar{q}_m| \to 0,$$

whence $\bar{p}_m \to \bar{r}$. Similarly, we obtain $\bar{q}_m \to \bar{r}$.

Now since $\bar{r} \in L[\bar{p}, \bar{q}] \subseteq B$, the function f is relatively continuous at $\bar{r}$ over B (by assumption). By the sequential criterion, then,

$$f(\bar{p}_m) \to f(\bar{r}) \text{ and } f(\bar{q}_m) \to f(\bar{r}).$$

Moreover, $f(\bar{p}_m) < c < f(\bar{q}_m)$ (for $\bar{p}_m \in P$ and $\bar{q}_m \in Q$). Letting $m \to +\infty$, we pass to limits (Chapter 3, §15, Corollary 1) and get

$$f(\bar{r}) \leq c \leq f(\bar{r}),$$

so that $\bar{r}$ *is neither in P nor in Q*, which is a contradiction. This completes the proof for a *convex* B.

The extension to polygon-connected sets is left as an exercise (see Problem 2 below). Thus all is proved. □

Note 2. In particular, the theorem applies if B is a globe or an interval.

Thus *continuity on an interval implies the Darboux property*. The converse *fails*, as we have noted. However, for *monotone* functions, we obtain the following theorem.

Theorem 2. *If a function $f\colon A \to E^1$ is monotone and has the Darboux property on a finite or infinite interval $(a, b) \subseteq A \subseteq E^1$, then it is continuous on (a, b).*

Proof. Seeking a contradiction, suppose f is discontinuous at some $p \in (a, b)$.

For definiteness, let $f\uparrow$ on (a, b). Then by Theorems 2 and 3 in §5, we have either $f(p^-) < f(p)$ or $f(p) < f(p^+)$ or both, *with no function values in between*.

On the other hand, since f has the Darboux property, the function values $f(x)$ for x in (a, b) *fill an entire interval* (see Note 1). Thus it is impossible for $f(p)$ to be the *only* function value between $f(p^-)$ and $f(p^+)$ unless f is *constant* near p, but then it is also continuous at p, which we excluded. This contradiction completes the proof.[3] □

Note 3. The theorem holds (with a similar proof) for *nonopen* intervals as well, but the continuity at the endpoints is *relative* (right at a, left at b).

Theorem 3. *If $f\colon A \to E^1$ is strictly monotone and continuous when restricted to a finite or infinite interval $B \subseteq A \subseteq E^1$, then its inverse f^{-1} has the same properties on the set $f[B]$ (itself an interval, by Note 1 and Theorem 1).*[4]

Proof. It is easy to see that f^{-1} is increasing (decreasing) if f is; the proof is left as an exercise. Thus f^{-1} is monotone on $f[B]$ if f is so on B. To prove the relative continuity of f^{-1}, we use Theorem 2, i.e., show that f^{-1} has the Darboux property on $f[B]$.

Thus let $f^{-1}(p) < c < f^{-1}(q)$ for some $p, q \in f[B]$. We look for an $r \in f[B]$ such that $f^{-1}(r) = c$, i.e., $r = f(c)$. Now since $p, q \in f[B]$, the numbers $f^{-1}(p)$ and $f^{-1}(q)$ are in B, *an interval*. Hence also the intermediate value c is in B; thus it belongs to the domain of f, and so the function value $f(c)$ *exists*. It thus suffices to put $r = f(c)$ to get the result. □

Examples.

(a) Define $f\colon E^1 \to E^1$ by

$$f(x) = x^n \text{ for a fixed } n \in N.$$

As f is continuous (being a *monomial*), it has the Darboux property on E^1. By Note 1, setting $B = [0, +\infty)$, we have $f[B] = [0, +\infty)$. (Why?) Also, f is strictly increasing on B. Thus by Theorem 3, *the inverse function f^{-1}* (i.e., *the nth root function*) *exists and is continuous on $f[B] = [0, +\infty)$*.

If n is *odd*, then f^{-1} has these properties on all of E^1, by a similar proof; thus $\sqrt[n]{x}$ exists for $x \in E^1$.

(b) Logarithmic functions. From the example in §5, we recall that the expo-

[3] More formally, if, say, $f(p) < f(p^+)$, let $f(p) < c < f(p^+) \le f(p')$, $p' \in (p, b)$. (Such a p' exists since $f\uparrow$, and $f(p^+) = \inf\{f(x) \mid p < x < b\}$; see §5, Theorem 1.) By the Darboux property, $f(x) = c$ for some $x \in (a, b)$, but this contradicts Theorem 2 in §5.

[4] We write "f" for "f restricted to B" as well; cf. also footnote 1 in §8.

nential function given by

$$F(x) = a^x \quad (a > 0)$$

is continuous and strictly monotone on E^1.[5] Its inverse, F^{-1}, is called the *logarithmic function to the base* a, denoted $\log_a$. By Theorem 3, *it is continuous and strictly monotone on* $F[E^1]$.

To fix ideas, let $a > 1$, so $F\uparrow$ and $(F^{-1})\uparrow$. By Note 1, $F[E^1]$ is an *interval* with endpoints p and r, where

$$p = \inf F[E^1] = \inf\{a^x \mid -\infty < x < +\infty\}$$

and

$$r = \sup F[E^1] = \sup\{a^x \mid -\infty < x < +\infty\}.$$

Now by Problem 14(iii) of §2 (with $q = 0$),

$$\lim_{x\to+\infty} a^x = +\infty \text{ and } \lim_{x\to-\infty} a^x = 0.$$

As $F\uparrow$, we use Theorem 1 in §5 to obtain

$$r = \sup a^x = \lim_{x\to+\infty} a^x = +\infty \text{ and } p = \lim_{x\to-\infty} a^x = 0.$$

Thus $F[E^1]$, i.e., the *domain of* $\log_a$, is the interval $(p, r) = (0, +\infty)$. It follows that $\log_a x$ *is uniquely defined for* x *in* $(0, +\infty)$; it is called the *logarithm of* x *to the base* a.

The *range* of $\log_a$ (i.e. of F^{-1}) is the same as the *domain of* F, i.e., E^1. Thus if $a > 1$, $\log_a x$ increases from $-\infty$ to $+\infty$ as x increases from 0 to $+\infty$. Hence

$$\lim_{x\to+\infty} \log_a x = +\infty \text{ and } \lim_{x\to0+} \log_a x = -\infty,$$

provided $a > 1$.

If $0 < a < 1$, the values of these limits are interchanged (since $F\downarrow$ in this case), but otherwise the results are the same.

If $a = e$, we write $\ln x$ or $\log x$ for $\log_a x$, and we call $\ln x$ the *natural logarithm* of x. Its inverse is, of course, the exponential $f(x) = e^x$, also written $\exp(x)$. Thus by definition, $\ln e^x = x$ and

$$x = \exp(\ln x) = e^{\ln x} \quad (0 < x < +\infty). \tag{1}$$

(c) The *power function* $g\colon (0, +\infty) \to E^1$ is defined by

$$g(x) = x^a \text{ for a fixed real } a.$$

[5] We exclude the case $a = 1$ here.

If $a > 0$, we also define $g(0) = 0$. For $x > 0$, we have

$$x^a = \exp(\ln x^a) = \exp(a \cdot \ln x).$$

Thus by the rules for composite functions (Theorem 3 and Corollary 2 in §2), the continuity of g on $(0, +\infty)$ follows from that of exponential and log functions. If $a > 0$, g is also continuous at 0. (Exercise!)

Problems on the Darboux Property and Related Topics

1. Prove Note 1.

1′. Prove Note 3.

1″. Prove continuity at 0 in Example (c).

2. Prove Theorem 1 for polygon-connected sets.
[Hint: If

$$B \supseteq \bigcup_{i=0}^{m-1} L[\bar{p}_i, \bar{p}_{i+1}]$$

with

$$f(\bar{p}_0) < c < f(\bar{p}_m),$$

show that for at least one i, either $c = f(\bar{p}_i)$ or $f(\bar{p}_i) < c < f(\bar{p}_{i+1})$. Then replace B in the theorem by the *convex* segment $L[\bar{p}_i, \bar{p}_{i+1}]$.]

3. Show that, if f is *strictly* increasing on $B \subseteq E$, then f^{-1} has the same property on $f[B]$, and both are one to one; similarly for decreasing functions.

4. For functions on $B = [a, b] \subset E^1$, Theorem 1 can be proved thusly: If

$$f(a) < c < f(b),$$

let

$$P = \{x \in B \mid f(x) < c\}$$

and put $r = \sup P$.

Show that $f(r)$ is neither greater nor less than c, and so necessarily $f(r) = c$.
[Hint: If $f(r) < c$, continuity at r implies that $f(x) < c$ on some $G_r(\delta)$ (§2, Problem 7), contrary to $r = \sup P$. (Why?)]

5. Continuing Problem 4, prove Theorem 1 *in all generality*, as follows. Define

$$g(t) = \bar{p} + t(\bar{q} - \bar{p}), \quad 0 \leq t \leq 1.$$

Then g is continuous (by Theorem 3 in §3), and so is the composite function $h = f \circ g$, on $[0, 1]$. By Problem 4, with $B = [0, 1]$, there is a $t \in (0, 1)$ with $h(t) = c$. Put $\bar{r} = g(t)$, and show that $f(\bar{r}) = c$.

6. Show that every equation of *odd* degree, of the form

$$f(x)=\sum_{k=0}^{n} a_k x^k = 0 \quad (n = 2m-1),$$

has at least one solution for x in E^1.
[Hint: Show that f takes both negative and positive values as $x \to -\infty$ or $x \to +\infty$; thus by the Darboux property, f must also take the intermediate value 0 for some $x \in E^1$.]

7. Prove that if the functions $f\colon A \to (0, +\infty)$ and $g\colon A \to E^1$ are both continuous, so also is the function $h\colon A \to E^1$ given by

$$h(x) = f(x)^{g(x)}.$$

[Hint: See Example (c)].

8. Using Corollary 2 in §2, and limit properties of the exponential and log functions, prove the "shorthand" Theorems 11–16 of §4.

8′. Find $\displaystyle\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{\sqrt{x}}$.

8″. Similarly, find a new solution of Problem 27 in Chapter 3, §15, reducing it to Problem 26.

9. Show that if $f\colon E^1 \to E^*$ has the Darboux property on B (e.g., if B is convex and f is relatively continuous on B) and if f is one to one on B, then f is necessarily strictly monotone on B.

10. Prove that if two real functions f, g are relatively continuous on $[a, b]$ $(a < b)$ and

$$f(x)g(x) > 0 \text{ for } x \in [a, b],$$

then the equation

$$(x-a)f(x) + (x-b)g(x) = 0$$

has a solution between a and b; similarly for the equation

$$\frac{f(x)}{x-a} + \frac{g(x)}{x-b} = 0 \quad (a, b \in E^1).$$

10′. Similarly, discuss the solutions of

$$\frac{2}{x-4} + \frac{9}{x-1} + \frac{1}{x-2} = 0.$$

§10. Arcs and Curves. Connected Sets

A deeper insight into continuity and the Darboux property can be gained by generalizing the notions of a convex set and polygon-connected set to obtain so-called *connected* sets.

I. As a first step, we consider *arcs* and *curves*.

Definition 1.

A set $A \subseteq (S, \rho)$ is called an *arc* iff A is a continuous image of a compact interval $[a, b] \subset E^1$, i.e., iff there is a continuous mapping

$$f\colon [a, b] \underset{\text{onto}}{\longrightarrow} A.$$

If, in addition, f is one to one, A is called a *simple* arc with *endpoints* $f(a)$ and $f(b)$.

If instead $f(a) = f(b)$, we speak of a *closed curve*.

A *curve* is a continuous image of *any* finite or infinite interval in E^1.

Corollary 1. *Each arc is a compact (hence closed and bounded) set (by Theorem 1 of* §8).

Definition 2.

A set $A \subseteq (S, \rho)$ is said to be *arcwise connected* iff every two points $p, q \in A$ are in some simple arc contained in A. (We then also say the p and q can be *joined* by an arc in A.)

Examples.

(a) Every closed line segment $L[\bar{a}, \bar{b}]$ in E^n (*or in any other normed space) is a simple arc (consider the map f in Example (1) of §8).

(b) Every *polygon*

$$A = \bigcup_{i=0}^{m-1} L[\bar{p}_i, \bar{p}_{i+1}]$$

is an arc (see Problem 18 in §8). It is a *simple* arc if the half-closed segments $L[\bar{p}_i, \bar{p}_{i+1})$ do not intersect and the points $\bar{p}_i$ are distinct, for then the map f in Problem 18 of §8 is one to one.

(c) It easily follows that *every polygon-connected set is also arcwise connected*; one only has to show that every polygon joining two points $\bar{p}_0, \bar{p}_m$ can be reduced to a *simple* polygon (not a self-intersecting one). See Problem 2.

However, the converse is false. For example, two discs in E^2 connected by a parabolic arc form together an *arcwise-* (but *not polygonwise-*) connected set.

(d) Let $f_1, f_2, \ldots, f_n$ be real continuous functions on an interval $I \subseteq E^1$. Treat them as components of a function $f\colon I \to E^n$,

$$f = (f_1, \ldots, f_n).$$

Then f is continuous by Theorem 2 in §3. Thus the image set $f[I]$ is a *curve* in E^n; it is an *arc* if I is a closed interval.

Introducing a parameter t varying over I, we obtain the *parametric equations* of the curve, namely,

$$x_k = f_k(t), \quad k = 1, 2, \ldots, n.$$

Then as t varies over I, the point $\bar{x} = (x_1, \ldots, x_n)$ describes the curve $f[I]$. This is the usual way of treating curves in E^n (*and C^n).

It is not hard to show that Theorem 1 in §9 holds also if B is only *arcwise* connected (see Problem 3 below). However, much more can be proved by introducing the general notion of a *connected* set. We do this next.

***II.** For this topic, we shall need Theorems 2–4 of Chapter 3, §12, and Problem 15 of Chapter 4, §2. The reader is advised to review them. In particular, we have the following theorem.

Theorem 1. *A function $f\colon (A, \rho) \to (T, \rho')$ is continuous on A iff $f^{-1}[B]$ is closed in (A, ρ) for each closed set $B \subseteq (T, \rho')$; similarly for open sets.*

Indeed, this is part of Problem 15 in §2 with (S, ρ) replaced by (A, ρ).

Definition 3.

A metric space (S, ρ) is said to be *connected* iff S is *not* the union $P \cup Q$ of any two nonvoid disjoint closed sets; it is *disconnected* otherwise.[1]

A *set* $A \subseteq (S, \rho)$ is called connected iff (A, ρ) is connected *as a subspace of* (S, ρ); i.e., iff A is not a union of two disjoint sets $P, Q \neq \emptyset$ that are closed (hence also open) *in* (A, ρ)*, as a subspace of* (S, ρ).

Note 1. By Theorem 4 of Chapter 3, §12, this means that

$$P = A \cap P_1 \text{ and } Q = A \cap Q_1$$

for some sets P_1, Q_1 that are closed *in* (S, ρ). Observe that, unlike compact sets, a set that is closed or open in (A, ρ) need not be closed or open in (S, ρ).

Examples.

(a′) $\emptyset$ is connected.

(b′) So is any one-point set $\{p\}$. (Why?)

[1] The term "closed" may be replaced by "open" here, for P and Q are open as well, *each being the complement of the other closed set*. Similarly, if they are open, they are both open and closed (briefly, "clopen").

(c′) Any *finite* set of *two or more* points is disconnected. (Why?)

Other examples are provided by the theorems that follow.

Theorem 2. *The only connected sets in* E^1 *are exactly all convex sets, i.e., finite and infinite intervals, including* E^1 *itself.*

Proof. The proof that such intervals are exactly *all* convex sets in E^1 is left as an exercise.

We now show that each connected set $A \subseteq E^1$ is convex, i.e., that $a, b \in A$ *implies* $(a, b) \subseteq A$.

Seeking a contradiction, suppose $p \notin A$ for some $p \in (a, b)$, $a, b \in A$. Let

$$P = A \cap (-\infty, p) \text{ and } Q = A \cap (p, +\infty).$$

Then $A = P \cup Q$, $a \in P$, $b \in Q$, and $P \cap Q = \emptyset$. Moreover, $(-\infty, p)$ and $(p, +\infty)$ are *open sets* in E^1. (Why?) Hence P and Q are *open in* A, each being the intersection of A with a set open in E^1 (see Note 1 above). As $A = P \cup Q$, with $P \cap Q = \emptyset$, it follows that A is *disconnected*. This shows that if A is connected in E^1, it *must* be convex.

Conversely, let A be convex in E^1. The proof that A is connected is an almost exact copy of the proof given for Theorem 1 of §9, so we only briefly sketch it here.[2]

If A were disconnected, then $A = P \cup Q$ for some disjoint sets $P, Q \neq \emptyset$, both closed in A. Fix any $p \in P$ and $q \in Q$. Exactly as in Theorem 1 of §9, select a contracting sequence of line segments (intervals) $[p_m, q_m] \subseteq A$ such that $p_m \in P$, $q_m \in Q$, and $|p_m - q_m| \to 0$, and obtain a point

$$r \in \bigcap_{m=1}^{\infty} [p_m, q_m] \subseteq A,$$

so that $p_m \to r$, $q_m \to r$, and $r \in A$. As the sets P and Q are *closed* in (A, ρ), Theorem 4 of Chapter 3, §16 shows that *both* P and Q must contain the common limit r of the sequences $\{p_m\} \subseteq P$ and $\{q_m\} \subseteq Q$. This is impossible, however, since $P \cap Q = \emptyset$, by assumption. This contradiction shows that A cannot be disconnected. Thus all is proved. □

Note 2. By the same proof, *any convex set in a normed space is connected*. In particular, E^n *and all other normed spaces are connected themselves*.[3]

Theorem 3. *If a function* $f\colon A \to (T, \rho')$ *with* $A \subseteq (S, \rho)$ *is relatively continuous on a connected set* $B \subseteq A$, *then* $f[B]$ *is a connected set in* (T, ρ').[4]

[2] Note that the same proof holds also for A *in any normed space*.

[3] See also Corollary 3 below (note that it *presupposes* Corollary 2, hence Theorem 2).

[4] Briefly, *any continuous image of a connected set is connected itself*.

Proof. By definition (§1), relative continuity on B becomes ordinary continuity *when* f *is restricted to* B. Thus we may treat f as a mapping *of* B *into* $f[B]$, replacing S and T by their subspaces B and $f[B]$.

Seeking a contradiction, suppose $f[B]$ is disconnected, i.e.,

$$f[B] = P \cup Q$$

for some disjoint sets P, $Q \neq \emptyset$ closed in $(f[B], \rho')$. Then by Theorem 1, with T replaced by $f[B]$, the sets $f^{-1}[P]$ and $f^{-1}[Q]$ are closed in (B, ρ). They also are nonvoid and disjoint (as are P and Q) and satisfy

$$B = f^{-1}[P \cup Q] = f^{-1}[P] \cup f^{-1}[Q]$$

(see Chapter 1, §§4–7, Problem 6). Thus B is disconnected, contrary to assumption. □

Corollary 2. *All arcs and curves are connected sets* (*by Definition* 2 *and Theorems* 2 *and* 3).

Lemma 1. *A set* $A \subseteq (S, \rho)$ *is connected iff any two points* $p, q \in A$ *are in some connected subset* $B \subseteq A$. *Hence any arcwise connected set is connected.*

Proof. Seeking a contradiction, suppose the condition stated in Lemma 1 holds but A is disconnected, so $A = P \cup Q$ for some disjoint sets $P \neq \emptyset$, $Q \neq \emptyset$, both closed in (A, ρ).

Pick any $p \in P$ and $q \in Q$. By assumption, p and q are in some connected set $B \subseteq A$. Treat (B, ρ) as a subspace of (A, ρ), and let

$$P' = B \cap P \text{ and } Q' = B \cap Q.$$

Then by Theorem 4 of Chapter 3, §12, P' and Q' are closed *in* B. Also, they are disjoint (for P and Q are) and nonvoid (for $p \in P'$, $q \in Q'$), and

$$B = B \cap A = B \cap (P \cup Q) = (B \cap P) \cup (B \cap Q) = P' \cup Q'.$$

Thus B is disconnected, contrary to assumption. This contradiction proves the lemma (the converse proof is trivial).

In particular, if A is arcwise connected, then any points p, q in A are in some *arc* $B \subseteq A$, a *connected set* by Corollary 2. Thus all is proved. □

Corollary 3. *Any convex or polygon-connected set* (*e.g.*, *a globe*) *in* E^n (*or in any other normed space*) *is arcwise connected*, *hence connected*.

Proof. Use Lemma 1 and Example (c) in part I of this section. □

Caution: The converse *fails*. *A connected set need not be arcwise connected, let alone polygon connected* (see Problem 17). However, we have the following theorem.

Theorem 4. *Every* open *connected set A in E^n (*or in another normed space) is also arcwise connected and even polygon connected.*

Proof. If $A = \emptyset$, this is "vacuously" true, so let $A \neq \emptyset$ and fix $\bar{a} \in A$.

Let P be the set of *all $\bar{p} \in A$ that can be joined with $\bar{a}$ by a polygon $K \subseteq A$.* Let $Q = A - P$. Clearly, $\bar{a} \in P$, so $P \neq \emptyset$. We shall show that P is *open*, i.e., that each $\bar{p} \in P$ is in a *globe* $G_{\bar{p}} \subseteq P$.

Thus we fix any $\bar{p} \in P$. As A is open and $\bar{p} \in A$, there certainly is a globe $G_{\bar{p}}$ contained *in* A. Moreover, as $G_{\bar{p}}$ is convex, each point $\bar{x} \in G_{\bar{p}}$ is joined with $\bar{p}$ by the line segment $L[\bar{x}, \bar{p}] \subseteq G_{\bar{p}}$. Also, as $\bar{p} \in P$, some polygon $K \subseteq A$ joins $\bar{p}$ with $\bar{a}$. Then

$$K \cup L[\bar{x}, \bar{p}]$$

is a polygon joining $\bar{x}$ and $\bar{a}$, and hence by definition $\bar{x} \in P$. Thus *each $\bar{x} \in G_{\bar{p}}$ is in P*, so that $G_{\bar{p}} \subseteq P$, as required, and P is open (also *open* in A *as a subspace*).

Next, we show that the set $Q = A - P$ is open as well. As before, if $Q \neq \emptyset$, fix any $\bar{q} \in Q$ and a globe $G_{\bar{q}} \subseteq A$, and show that $G_{\bar{q}} \subseteq Q$. Indeed, if some $\bar{x} \in G_{\bar{q}}$ were *not* in Q, it would be in P, and thus it would be joined with $\bar{a}$ (fixed above) by a polygon $K \subseteq A$. Then, however, $\bar{q}$ *itself could be so joined* by the polygon

$$L[\bar{q}, \bar{x}] \cup K,$$

implying that $\bar{q} \in P$, *not* $\bar{q} \in Q$. This shows that $G_{\bar{q}} \subset Q$ indeed, as claimed.

Thus $A = P \cup Q$ with P, Q disjoint and *open* (hence *clopen*) in A. The connectedness of A then implies that $Q = \emptyset$. (P is *not* empty, as has been noted.) Hence $A = P$. By the definition of P, then, each point $\bar{b} \in A$ can be joined to $\bar{a}$ by a polygon. As $\bar{a} \in A$ was arbitrary, A is polygon connected. □

Finally, we obtain a stronger version of the *intermediate value theorem.*

Theorem 5. *If a function $f\colon A \to E^1$ is relatively continuous on a connected set $B \subseteq A \subseteq (S, \rho)$, then f has the Darboux property on B.*

In fact, by Theorems 3 and 2, $f[B]$ is a *connected set in E^1*, i.e., an *interval.* This, however, implies the Darboux property.

Problems on Arcs, Curves, and Connected Sets

1. Discuss Examples (a) and (b) in detail. In particular, verify that $L[\bar{a}, \bar{b}]$ is a *simple* arc. (Show that the map f in Example (1) of §8 is *one to one.*)

2. Show that each polygon

$$K = \bigcup_{i=0}^{m-1} L[\bar{p}_i, \bar{p}_{i+1}]$$

can be reduced to a *simple* polygon P ($P \subseteq K$) joining p_0 and p_m.
[Hint: First, show that if two line segments have two or more common points, they lie in one line. Then use induction on the number m of segments in K. Draw a diagram in E^2 as a guide.]

3. Prove Theorem 1 of §9 for an *arcwise connected* $B \subseteq (S, \rho)$.
[Hint: Proceed as in Problems 4 and 5 in §9, replacing g by some continuous map $f\colon [a, b] \underset{\text{onto}}{\longrightarrow} B$.]

4. Define f as in Example (f) of §1. Let

$$G_{ab} = \{(x, y) \in E^2 \mid a \leq x \leq b,\ y = f(x)\}.$$

(G_{ab} is the *graph of f over* $[a, b]$.) Prove the following:

(i) If $a > 0$, then G_{ab} is a simple arc in E^2.

(ii) If $a \leq 0 \leq b$, G_{ab} is not even arcwise connected.

[Hints: (i) Prove that f is continuous on $[a, b]$, $a > 0$, using the continuity of the sine function. Then use Problem 16 in §2, restricting f to $[a, b]$.
(ii) For a contradiction, assume $\bar{0}$ is joined by a simple arc to some $\bar{p} \in G_{ab}$.]

5. Show that each arc is a continuous image of $[0, 1]$.
[Hint: First, show that any $[a, b] \subseteq E^1$ is such an image. Then use a suitable composite mapping.]

***6.** Prove that a function $f\colon B \to E^1$ on a compact set $B \subseteq E^1$ *must* be continuous if its graph,

$$\{(x, y) \in E^2 \mid x \in B,\ y = f(x)\},$$

is a compact set (e.g., an *arc*) in E^2.
[Hint: Proceed as in the proof of Theorem 3 of §8.]

***7.** Prove that A is connected iff there is no continuous map

$$f\colon A \underset{\text{onto}}{\longrightarrow} \{0, 1\}.[5]$$

[Hint: If there is such a map, Theorem 1 shows that A is disconnected. (Why?) Conversely, if $A = P \cup Q$ (P, Q as in Definition 3), put $f = 0$ on P and $f = 1$ on Q. Use again Theorem 1 to show that f so defined is continuous on A.]

***8.** Let $B \subseteq A \subseteq (S, \rho)$. Prove that B is connected in S iff it is connected in (A, ρ).

***9.** Suppose that no two of the sets A_i ($i \in I$) are disjoint. Prove that if all A_i are connected, so is $A = \bigcup_{i \in I} A_i$.
[Hint: If not, let $A = P \cup Q$ (P, Q as in Definition 3). Let $P_i = A_i \cap P$ and $Q_i = A_i \cap Q$, so $A_i = P_i \cup Q_i$, $i \in I$.

[5] That is, onto a *two-point* set $\{0\} \cup \{1\}$.

At least one of the P_i, Q_i must be $\emptyset$ (why?); say, $Q_j = \emptyset$ for some $j \in I$. Then $(\forall i)\ Q_i = \emptyset$, for $Q_i \neq \emptyset$ implies $P_i = \emptyset$, whence

$$A_i = Q_i \subseteq Q \Longrightarrow A_i \cap A_j = \emptyset \text{ (since } A_j \subseteq P),$$

contrary to our assumption. Deduce that $Q = \bigcup_i Q_i = \emptyset$. (Contradiction!)]

***10.** Prove that if $\{A_n\}$ is a finite or infinite sequence of connected sets and if

$$(\forall n) \quad A_n \cap A_{n+1} \neq \emptyset,$$

then

$$A = \bigcup_n A_n$$

is connected.

[Hint: Let $B_n = \bigcup_{k=1}^n A_k$. Use Problem 9 and induction to show that the B_n are connected and no two are disjoint. Verify that $A = \bigcup_n B_n$ and apply Problem 9 to the sets B_n.]

***11.** Given $p \in A$, $A \subseteq (S, \rho)$, let A_p denote the union of all *connected* subsets of A that contain p (one of them is $\{p\}$); A_p is called the *p-component* of A. Prove that

(i) A_p is connected (use Problem 9);

(ii) A_p is not contained in any other connected set $B \subseteq A$ with $p \in B$;

(iii) $(\forall p, q \in A)\ A_p \cap A_q = \emptyset$ iff $A_p \neq A_q$; and

(iv) $A = \bigcup\{A_p \mid p \in A\}$.

[Hint for (iii): If $A_p \cap A_q \neq \emptyset$ and $A_p \neq A_q$, then $B = A_p \cup A_q$ is a connected set *larger* than A_p, contrary to (ii).]

***12.** Prove that if A is connected, so is its closure (Chapter 3, §16, Definition 1), and so is any set D such that $A \subseteq D \subseteq \bar{A}$.

[Hints: First show that D is the "least" closed set *in* (D, ρ) that contains A (Problem 11 in Chapter 3, §16 and Theorem 4 of Chapter 3, §12). Next, seeking a contradiction, let $D = P \cup Q$, $P \cap Q = \emptyset$, $P, Q \neq \emptyset$, clopen *in* D. Then

$$A = (A \cap P) \cup (A \cap Q)$$

proves A *disconnected*, for if $A \cap P = \emptyset$, say, then $A \subseteq Q \subset D$ (why?), contrary to the minimality of D; similarly for $A \cap Q = \emptyset$.]

***13.** A set is said to be *totally disconnected* iff its only connected subsets are one-point sets and $\emptyset$.

Show that R (the rationals) has this property in E^1.

***14.** Show that any *discrete* space is totally disconnected (see Problem 13).

***15.** From Problems 11 and 12 deduce that each component A_p is *closed* $(A_p = \overline{A_p})$.

***16.** Prove that a set $A \subseteq (S, \rho)$ is disconnected iff $A = P \cup Q$, with $P, Q \neq \emptyset$, and each of P, Q disjoint from the closure of the other: $P \cap \overline{Q} = \emptyset = \overline{P} \cap Q$.
[Hint: By Problem 12, the closure of P *in* (A, ρ) (i.e., the least closed set *in* (A, ρ) that contains P) is

$$A \cap \overline{P} = (P \cup Q) \cap \overline{P} = (P \cap \overline{P}) \cup (Q \cap \overline{P}) = P \cup \emptyset = P,$$

so P is closed in A; similarly for Q. Prove the converse in the same manner.]

***17.** Give an example of a connected set that is not arcwise connected.
[Hint: The set G_{0b} $(a = 0)$ in Problem 4 is the closure of $G_{0b} - \{\bar{0}\}$ (verify!), and the latter is connected (why?); hence so is G_{0b} by Problem 12.]

*§11. Product Spaces. Double and Iterated Limits

Given two metric spaces (X, ρ_1) and (Y, ρ_2), we may consider the Cartesian product $X \times Y$, *suitably metrized*. Two metrics for $X \times Y$ are suggested in Problem 10 in Chapter 3, §11. We shall adopt the first of them as follows.

Definition 1.

By the *product* of two metric spaces (X, ρ_1) and (Y, ρ_2) is meant the space $(X \times Y, \rho)$, where the metric ρ is defined by

$$\rho((x, y), (x', y')) = \max\{\rho_1(x, x'), \rho_2(y, y')\} \tag{1}$$

for $x, x' \in X$ and $y, y' \in Y$.

Thus *the distance between* (x, y) *and* (x', y') *is the larger of the two distances*

$$\rho_1(x, x') \text{ in } X \text{ and } \rho_2(y, y') \text{ in } Y.$$

The verification that ρ in (1) is, indeed, a metric is left to the reader. We now obtain the following theorem.

Theorem 1.

(i) *A globe* $G_{(p, q)}(\varepsilon)$ *in* $(X \times Y, \rho)$ *is the Cartesian product of the corresponding* ε*-globes in* X *and* Y,

$$G_{(p, q)}(\varepsilon) = G_p(\varepsilon) \times G_q(\varepsilon).$$

(ii) *Convergence of sequences* $\{(x_m, y_m)\}$ *in* $X \times Y$ *is componentwise. That is, we have*

$$(x_m, y_m) \to (p, q) \text{ in } X \times Y \text{ iff } x_m \to p \text{ in } X \text{ and } y_m \to q \text{ in } Y.$$

Again, the easy proof is left as an exercise.

In this connection, recall that by Theorem 2 of Chapter 3, §15, *convergence in E^2 is componentwise as well*, even though the standard metric in E^2 is *not* the product metric (1); it is rather the metric (ii) of Problem 10 in Chapter 3, §11. We might have adopted this second metric for $X \times Y$ as well. Then part (i) of Theorem 1 would fail, but *part* (ii) *would still follow* by making

$$\rho_1(x_m, p) < \frac{\varepsilon}{\sqrt{2}} \text{ and } \rho_2(y_m, q) < \frac{\varepsilon}{\sqrt{2}}.$$

It follows that, as far as convergence is concerned, *the two choices of ρ are equivalent*.

Note 1. More generally, two metrics for a space S are said to be *equivalent* iff exactly the same sequences converge (to the same limits) under both metrics. Then also all function limits are the same since they reduce to sequential limits, by Theorem 1 of §2; similarly for such notions as continuity, compactness, completeness, closedness, openness, etc.

In view of this, we shall often call $X \times Y$ a *product space* (*in the wider sense*) even if its metric is not the ρ of formula (1) but equivalent to it. In this sense, E^2 *is* the product space $E^1 \times E^1$, and $X \times Y$ is its generalization.

Various ideas valid in E^2 extend quite naturally to $X \times Y$. Thus functions defined on a set $A \subseteq X \times Y$ may be treated as *functions of two variables* x, y such that $(x, y) \in A$. Given $(p, q) \in X \times Y$, we may consider ordinary or relative limits at (p, q), e.g., limits over a path

$$B = \{(x, y) \in X \times Y \mid y = q\}$$

(briefly called the "*line* $y = q$"). In this case, y remains fixed $(y = q)$ while $x \to p$; we then speak of limits and continuity *in one variable* x, as opposed to those *in both variables jointly*, i.e., the ordinary limits (cf. §3, part IV).

Some other kinds of limits are to be defined below. For simplicity, we consider only functions $f\colon (X \times Y) \to (T, \rho')$ defined on *all* of $X \times Y$. If confusion is unlikely, we write ρ for *all* metrics involved (such as ρ' in T). Below, p and q always denote cluster points of X and Y, respectively (this justifies the "lim" notation). Of course, our definitions apply in particular to E^2 as the simplest special case of $X \times Y$.

Definition 2.

A function $f\colon (X \times Y) \to (T, \rho')$ is said to have the *double limit* $s \in T$ at (p, q), denoted

$$s = \lim_{\substack{x \to p \\ y \to q}} f(x, y),$$

iff for each $\varepsilon > 0$, there is a $\delta > 0$ such that $f(x, y) \in G_s(\varepsilon)$ whenever

$x \in G_{\neg p}(\delta)$ and $y \in G_{\neg q}(\delta)$. In symbols,

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x \in G_{\neg p}(\delta))\ (\forall y \in G_{\neg q}(\delta)) \quad f(x, y) \in G_s(\varepsilon). \tag{2}$$

Observe that this is the *relative* limit over the path

$$D = (X - \{p\}) \times (Y - \{q\})$$

excluding the two "lines" $x = p$ and $y = q$. If f were *restricted* to D, this would coincide with the *ordinary* nonrelative limit (see §1), denoted

$$s = \lim_{(x, y) \to (p, q)} f(x, y),$$

where only the *point* (p, q) is excluded. Then we would have

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall (x, y) \in G_{\neg (p,q)}(\delta)) \quad f(x, y) \in G_s(\varepsilon). \tag{3}$$

Now consider limits in *one* variable, say,

$$\lim_{y \to q} f(x, y) \text{ with } x \textit{ fixed}.$$

If this limit exists for *each* choice of x from some set $B \subseteq X$, it defines a *function*

$$g\colon B \to T$$

with value

$$g(x) = \lim_{y \to q} f(x, y), \quad x \in B.$$

This means that

$$(\forall x \in B)\ (\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall y \in G_{\neg q}(\delta)) \quad \rho(g(x), f(x, y)) < \varepsilon. \tag{4}$$

Here, in general, δ depends on *both* ε and x. However, in some cases (resembling uniform continuity), *one and the same* δ (depending *on* ε *only*) fits *all* choices of x from B. This suggests the following definition.

Definition 3.

With the previous notation, suppose

$$\lim_{y \to q} f(x, y) = g(x) \text{ exists for each } x \in B\ (B \subseteq X).$$

We say that this limit is *uniform in x (on B)*, and we write

$$\text{“}g(x) = \lim_{y \to q} f(x, y)\ (\textit{uniformly for } x \in B)\text{,”}$$

iff for each $\varepsilon > 0$, there is a $\delta > 0$ such that $\rho(g(x), f(x, y)) < \varepsilon$ for *all* $x \in B$ and *all* $y \in G_{\neg q}(\delta)$. In symbols,

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x \in B)\ (\forall y \in G_{\neg q}(\delta)) \quad \rho(g(x), f(x, y)) < \varepsilon. \tag{5}$$

Usually, the set B in formulas (4) and (5) is a *deleted neighborhood of* p in X, e.g.,

$$B = G_{\neg p}(r), \text{ or } B = X - \{p\}.$$

Assume (4) for such a B, so

$$\lim_{y\to q} f(x, y) = g(x) \text{ exists for each } x \in B.$$

If, in addition,

$$\lim_{x\to p} g(x) = s$$

exists, we call s *the iterated limit of* f *at* (p, q) (*first in* y*, then in* x), denoted

$$\lim_{x\to p}\lim_{y\to q} f(x, y).$$

This limit is obtained by *first* letting $y \to q$ (with x fixed) and *then* letting $x \to p$. Quite similarly, we define

$$\lim_{y\to q}\lim_{x\to p} f(x, y).$$

In general, the two iterated limits (if they exist) are *different*, and their existence does not imply that of the double limit (2), let alone (3), nor does it imply the equality of all these limits. (See Problems 4ff below.) However, we have the following theorem.

Theorem 2 (Osgood)**.** *Let* (T, ρ') *be complete. Assume the existence of the following limits of the function* $f\colon X \times Y \to T$:

(i) $\lim_{y\to q} f(x, y) = g(x)$ (*uniformly for* $x \in X - \{p\}$) *and*

(ii) $\lim_{x\to p} f(x, y) = h(y)$ *for* $y \in Y - \{q\}$.[1]

Then the double limit and the two iterated limits of f *at* (p, q) *exist and all three coincide.*

Proof. Let $\varepsilon > 0$. By our assumption (i), there is a $\delta > 0$ such that

$$(\forall\, x \in X - \{p\})\ (\forall\, y \in G_{\neg q}(\delta))\quad \rho(g(x), f(x,y)) < \frac{\varepsilon}{4} \quad \text{(cf. (5))}. \tag{5'}$$

Now take any y', $y'' \in G_{\neg q}(\delta)$. By assumption (ii), there is an $x' \in X - \{p\}$ so close to p that

$$\rho(h(y'), f(x', y')) < \frac{\varepsilon}{4} \text{ and } \rho(h(y''), f(x', y'')) < \frac{\varepsilon}{4}. \text{ (Why?)}$$

[1] Actually, it suffices to assume the existence of the limits (i) and (ii) for x *in some* $G_{\neg p}(r)$ and y *in some* $G_{\neg q}(r)$. Of course, it does not matter *which* of the two limits is uniform.

Hence, using (5′) and the triangle law (repeatedly), we obtain for such y', y''

$$\begin{aligned}\rho(h(y'),\, h(y'')) &\le \rho(h(y'),\, f(x',\, y')) + \rho(f(x',\, y'),\, g(x'))\\ &\quad + \rho(g(x'),\, f(x', y'')) + \rho(f(x',\, y''),\, h(y''))\\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

It follows that the function h *satisfies the Cauchy criterion of Theorem* 2 *in* §2. (It *does* apply since T is complete.) Thus $\lim_{y\to q} h(y)$ *exists*, and, by assumption (ii), it equals $\lim\limits_{y\to q}\lim\limits_{x\to p} f(x,\, y)$ (which therefore *exists*).

Let then $H = \lim\limits_{y\to q} h(y)$. With δ as above, fix some $y_0 \in G_{\neg q}(\delta)$ so close to q that

$$\rho(h(y_0),\, H) < \frac{\varepsilon}{4}.$$

Also, using assumption (ii), choose a $\delta' > 0$ ($\delta' \le \delta$) such that

$$\rho(h(y_0),\, f(x,\, y_0)) < \frac{\varepsilon}{4} \quad \text{for } x \in G_{\neg p}(\delta').$$

Combining with (5′), obtain $(\forall\, x \in G_{\neg p}(\delta'))$

$$\rho(H,\, g(x)) \le \rho(H,\, h(y_0)) + \rho(h(y_0),\, f(x,\, y_0)) + \rho(f(x,\, y_0),\, g(x)) < \frac{3\varepsilon}{4}. \quad (6)$$

Thus

$$(\forall\, x \in G_{\neg p}(\delta')) \quad \rho(H,\, g(x)) < \varepsilon.$$

Hence $\lim_{x\to p} g(x) = H$, i.e., the second iterated limit, $\lim\limits_{x\to p}\lim\limits_{y\to q} f(x,\, y)$, likewise exists and equals H.

Finally, with the same $\delta' \le \delta$, we combine (6) and (5′) to obtain

$(\forall\, x \in G_{\neg p}(\delta'))$ $(\forall\, y \in G_{\neg q}(\delta'))$

$$\rho(H,\, f(x,\, y)) \le \rho(H,\, g(x)) + \rho(g(x),\, f(x,\, y)) < \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

Hence the double limit (2) also exists and equals H. □

Note 2. The same proof works also *with* f *restricted to* $(X-\{p\})\times(Y-\{q\})$ so that the "lines" $x = p$ and $y = q$ *are excluded from* D_f. In this case, formulas (2) and (3) mean the same; i.e.,

$$\lim_{\substack{x\to p\\ y\to q}} f(x,\, y) = \lim_{(x,\, y)\to(p,\, q)} f(x,\, y).$$

Note 3. In Theorem 2, we may take E^* (suitably metrized) for X or Y or T. Then the theorem also applies to limits at $\pm\infty$, and infinite limits. We may also take $X = Y = N \cup \{+\infty\}$ (the naturals together with $+\infty$), with the same E^*-metric, and consider limits at $p = q = +\infty$. Moreover, by Note 2, we may restrict f to $N \times N$, so that $f\colon N \times N \to T$ becomes a *double sequence*

(Chapter 1, §9). Writing m and n for x and y, and u_{mn} for $f(x, y)$, we then obtain *Osgood's theorem for double sequences* (also called the *Moore–Smith theorem*) as follows.

Theorem 2′. *Let $\{u_{mn}\}$ be a double sequence in a complete space (T, ρ'). If*

$$\lim_{n\to\infty} u_{mn} = q_m \text{ exists for each } m$$

and if

$$\lim_{m\to\infty} u_{mn} = p_n \text{ (uniformly in } n\text{) likewise exists,}$$

then the double limit and the two iterated limits of $\{u_{mn}\}$ exist and

$$\lim_{\substack{m\to\infty \\ n\to\infty}} u_{mn} = \lim_{n\to\infty} \lim_{m\to\infty} u_{mn} = \lim_{m\to\infty} \lim_{n\to\infty} u_{mn}.$$

Here the assumption that $\lim_{m\to\infty} u_{mn} = p_n$ (uniformly in n) means, by (5), that

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, n)\ (\forall\, m > k) \quad \rho(u_{mn}, p_n) < \varepsilon. \tag{7}$$

Similarly, the statement "$\lim\limits_{\substack{m\to\infty \\ n\to\infty}} u_{mn} = s$" (see (2)) is tantamount to

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, m, n > k) \quad \rho(u_{mn}, s) < \varepsilon. \tag{8}$$

Note 4. Given any sequence $\{x_m\} \subseteq (S, \rho)$, we may consider the double limit $\lim\limits_{\substack{m\to\infty \\ n\to\infty}} \rho(x_m, x_n)$ in E^1. By using (8), one easily sees that

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \rho(x_m, x_n) = 0$$

iff

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, m, n > k) \quad \rho(x_m, x_n) < \varepsilon,$$

i.e., *iff $\{x_m\}$ is a Cauchy sequence*. Thus *Cauchy sequences are those for which* $\lim\limits_{\substack{m\to\infty \\ n\to\infty}} \rho(x_m, x_n) = 0$.

Theorem 3. *In every metric space (S, ρ), the metric $\rho\colon (S \times S) \to E^1$ is a continuous function on the product space $S \times S$.*

Proof. Fix any $(p, q) \in S \times S$. By Theorem 1 of §2, ρ is continuous at (p, q) iff

$$\rho(x_m, y_m) \to \rho(p, q) \text{ whenever } (x_m, y_m) \to (p, q),$$

i.e., whenever $x_m \to p$ and $y_m \to q$. However, this follows by Theorem 4 in Chapter 3, §15. Thus continuity is proved. □

Problems on Double Limits and Product Spaces

1. Prove Theorem 1(i). Prove Theorem 1(ii) for *both* choices of ρ, as suggested.

2. Formulate Definitions 2 and 3 for the cases

(i) $p = q = s = +\infty$;

(ii) $p = +\infty$, $q \in E^1$, $s = -\infty$;

(iii) $p \in E^1$, $q = s = -\infty$; and

(iv) $p = q = s = -\infty$.

3. Prove Theorem 2′ from Theorem 2 *using Theorem 1 of* §2. Give a direct proof as well.

4. Define $f\colon E^2 \to E^1$ by

$$f(x, y) = \frac{xy}{x^2 + y^2} \text{ if } (x, y) \neq (0, 0), \text{ and } f(0, 0) = 0;$$

see §1, Example (g). Show that

$$\lim_{y\to 0} \lim_{x\to 0} f(x, y) = 0 = \lim_{x\to 0} \lim_{y\to 0} f(x, y),$$

but

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x, y) \text{ does not exist.}$$

Explain the apparent failure of Theorem 2.

4′. Define $f\colon E^2 \to E^1$ by

$$f(x, y) = 0 \text{ if } xy = 0 \text{ and } f(x, y) = 1 \text{ otherwise.}$$

Show that f satisfies Theorem 2 at $(p, q) = (0, 0)$, but

$$\lim_{(x, y)\to(p, q)} f(x, y)$$

does not exist.

5. Do Problem 4, with f defined as in Problems 9 and 10 of §3.

6. Define f as in Problem 11 of §3. Show that for (c), we have

$$\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{\substack{x\to 0\\ y\to 0}} f(x, y) = \lim_{x\to 0} \lim_{y\to 0} f(x, y) = 0,$$

but $\lim_{y\to 0} \lim_{x\to 0} f(x, y)$ does not exist; for (d),

$$\lim_{y\to 0} \lim_{x\to 0} f(x, y) = 0,$$

but the iterated limits do not exist; and for (e), $\lim_{(x,\,y)\to(0,\,0)} f(x,\,y)$ fails to exist, but

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,\,y) = \lim_{y\to 0}\lim_{x\to 0} f(x,\,y) = \lim_{x\to 0}\lim_{y\to 0} f(x,\,y) = 0.$$

Give your comments.

7. Find (if possible) the ordinary, the double, and the iterated limits of f at $(0,\,0)$ assuming that $f(x,\,y)$ is given by one of the expressions below, and f is defined at those points of E^2 where the expression has sense.

(i) $\dfrac{x^2}{x^2+y^2}$; (ii) $\dfrac{y\sin xy}{x^2+y^2}$;

(iii) $\dfrac{x+2y}{x-y}$; (iv) $\dfrac{x^3y}{x^6+y^2}$;

(v) $\dfrac{x^2-y^2}{x^2+y^2}$; (vi) $\dfrac{x^5+y^4}{(x^2+y^2)^2}$;

(vii) $\dfrac{y+x\cdot 2^{-y^2}}{4+x^2}$; (viii) $\dfrac{\sin xy}{\sin x\cdot\sin y}$.

8. Solve Problem 7 with x and y tending to $+\infty$.

9. Consider the sequence u_{mn} in E^1 defined by

$$u_{mn} = \frac{m+2n}{m+n}.$$

Show that

$$\lim_{m\to\infty}\lim_{n\to\infty} u_{mn} = 2 \text{ and } \lim_{n\to\infty}\lim_{m\to\infty} u_{mn} = 1,$$

but the double limit fails to exist. What is wrong here? (See Theorem 2′.)

10. Prove Theorem 2, with (i) replaced by the weaker assumption ("*subuniform limit*")

$$(\forall\,\varepsilon>0)\ (\exists\,\delta>0)\ (\forall\,x\in G_{\neg p}(\delta))\ (\forall\,y\in G_{\neg q}(\delta))\quad \rho(g(x),\,f(x,\,y))<\varepsilon$$

and with iterated limits defined by

$$s = \lim_{x\to p}\lim_{y\to q} f(x,\,y)$$

iff $(\forall\,\varepsilon>0)$

$$(\exists\,\delta'>0)\ (\forall\,x\in G_{\neg p}(\delta'))\ (\exists\,\delta''_x>0)\ (\forall\,y\in G_{\neg q}(\delta''_x))\quad \rho(f(x,\,y),\,s)<\varepsilon.$$

11. Does the continuity of f on $X \times Y$ imply the existence of (i) iterated limits? (ii) the double limit?
[Hint: See Problem 6.]

12. Show that the standard metric in E^1 is equivalent to ρ' of Problem 7 in Chapter 3, §11.

13. Define products of n spaces and prove Theorem 1 for such product spaces.

14. Show that the standard metric in E^n is equivalent to the product metric for E^n treated as a product of n spaces E^1. Solve a similar problem for C^n.
[Hint: Use Problem 13.]

15. Prove that $\{(x_m, y_m)\}$ is a Cauchy sequence in $X \times Y$ iff $\{x_m\}$ and $\{y_m\}$ are Cauchy. Deduce that $X \times Y$ is complete iff X and Y are.

16. Prove that $X \times Y$ is compact iff X and Y are.
[Hint: See the proof of Theorem 2 in Chapter 3, §16, for E^2.]

17. (i) Prove the uniform continuity of *projection maps* P_1 and P_2 on $X \times Y$, given by $P_1(x, y) = x$ and $P_2(x, y) = y$.

(ii) Show that for each open set G in $X \times Y$, $P_1[G]$ is open in X and $P_2[G]$ is open in Y.
[Hint: Use Corollary 1 of Chapter 3, §12.]

(iii) Disprove (ii) for *closed* sets by a counterexample.
[Hint: Let $X \times Y = E^2$. Let G be the hyperbola $xy = 1$. Use Theorem 4 of Chapter 3, §16 to prove that G is closed.]

18. Prove that if $X \times Y$ is connected, so are X and Y.
[Hint: Use Theorem 3 of §10 and the projection maps P_1 and P_2 of Problem 17.]

19. Prove that if X and Y are connected, so is $X \times Y$ under the product metric.
[Hint: Using suitable continuous maps and Theorem 3 in §10, show that any two "lines" $x = p$ and $y = q$ are connected sets in $X \times Y$. Then use Lemma 1 and Problem 10 in §10.]

20. Prove Theorem 2 under the weaker assumptions stated in footnote 1.

21. Prove the following:

(i) If

$$g(x) = \lim_{y \to q} f(x, y) \text{ and } H = \lim_{\substack{x \to p \\ y \to q}} f(x, y)$$

exist for $x \in G_{\neg p}(r)$ and $y \in G_{\neg q}(r)$, then

$$\lim_{x \to p} \lim_{y \to q} f(x, y) = H.$$

(ii) If the double limit and one iterated limit exist, they are *necessarily* equal.

22. In Theorem 2, add the assumptions

$$h(y) = f(p, y) \quad \text{for } y \in Y - \{q\}$$

and

$$g(x) = f(x, q) \quad \text{for } x \in X - \{p\}.$$

Then show that

$$\lim_{(x, y) \to (p, q)} f(x, y)$$

exists and equals the double limits.

[Hint: Show that here (5) holds also for $x = p$ and $y \in G_{\neg q}(\delta)$ and for $y = q$ and $x \in G_{\neg p}(\delta)$.]

23. From Problem 22 prove that a function $f\colon (X \times Y) \to T$ is continuous at (p, q) *if*

$$f(p, y) = \lim_{x \to p} f(x, y) \text{ and } f(x, q) = \lim_{y \to q} f(x, y)$$

for (x, y) in some $G_{(p, q)}(\delta)$, and at least one of these limits is uniform.

§12. Sequences and Series of Functions

I. Let

$$f_1, f_2, \ldots, f_m, \ldots$$

be a sequence of mappings from a common domain A into a metric space (T, ρ').[1] For each (fixed) $x \in A$, the function values

$$f_1(x), f_2(x), \ldots, f_m(x), \ldots$$

form a *sequence of points* in the range space (T, ρ'). Suppose this sequence converges for each x in a set $B \subseteq A$. Then we can define a function $f\colon B \to T$ by setting

$$f(x) = \lim_{m \to \infty} f_m(x) \quad \text{for all } x \in B.$$

This means that

$$(\forall \varepsilon > 0)\ (\forall x \in B)\ (\exists k)\ (\forall m > k) \quad \rho'(f_m(x), f(x)) < \varepsilon. \tag{1}$$

Here k depends not only on ε but *also on* x, since each x yields a *different* sequence $\{f_m(x)\}$. However, in some cases (resembling uniform continuity), k

[1] We briefly denote such a sequence by $f_m\colon A \to (T, \rho')$.

depends *on ε only*; i.e., given $\varepsilon > 0$, *one and the same k* fits *all x* in B. In symbols, this is indicated by changing the order of quantifiers, namely,

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, x \in B)\ (\forall\, m > k) \quad \rho'(f_m(x),\, f(x)) < \varepsilon. \tag{2}$$

Of course, (2) *implies* (1), but the converse fails (see examples below). This suggests the following definitions.

Definition 1.

With the above notation, we call f the *pointwise limit* of a sequence of functions f_m on a set B ($B \subseteq A$) iff

$$f(x) = \lim_{m\to\infty} f_m(x) \text{ for all } x \text{ in } B;$$

i.e., formula (1) holds. We then write

$$f_m \to f \ (\textit{pointwise}) \ \textit{on}\ B.$$

In case (2), we call the limit *uniform* (on B) and write

$$f_m \to f \ (\textit{uniformly}) \ \textit{on}\ B.$$

II. If the f_m are real, complex, or vector valued (§3), we can also define $s_m = \sum_{k=1}^{m} f_k$ (= sum of the first m functions) for each m, so

$$(\forall\, x \in A)\ (\forall\, m) \quad s_m(x) = \sum_{k=1}^{m} f_k(x).$$

The s_m form a new sequence of functions on A. The *pair* of sequences

$$(\{f_m\},\, \{s_m\})$$

is called the (infinite) *series* with *general term* f_m; s_m is called its *mth partial sum*. The series is often denoted by symbols like $\sum f_m$, $\sum f_m(x)$, etc.

Definition 2.

The series $\sum f_m$ on A is said to *converge* (pointwise or uniformly) to a function f on a set $B \subseteq A$ iff the sequence $\{s_m\}$ of its partial sums does as well.

We then call f the *sum* of the series and write

$$f(x) = \sum_{k=1}^{\infty} f_k(x) \text{ or } f = \sum_{m=1}^{\infty} f_m = \lim s_m$$

(pointwise or uniformly) on B.

Note that series of *constants*, $\sum c_m$, may be treated as series of *constant functions* f_m, with $f_m(x) = c_m$ for $x \in A$.

If the range space is E^1 or E^*, we also consider *infinite* limits,

$$\lim_{m\to\infty} f_m(x) = \pm\infty.$$

However, a *series* for which

$$\sum_{m=1}^{\infty} f_m = \lim s_m$$

is infinite for some x is regarded as *divergent* (i.e., *not* convergent) at that x.

III. Since convergence of series reduces to that of *sequences* $\{s_m\}$, we shall first of all consider sequences. The following is a simple and useful test for uniform convergence of sequences $f_m \colon A \to (T, \rho')$.

Theorem 1. *Given a sequence of functions $f_m \colon A \to (T, \rho')$, let $B \subseteq A$ and*

$$Q_m = \sup_{x\in B} \rho'(f_m(x), f(x)).$$

Then $f_m \to f$ (uniformly on B) iff $Q_m \to 0$.

Proof. If $Q_m \to 0$, then by definition

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, m > k) \quad Q_m < \varepsilon.$$

However, Q_m is an *upper bound* of all distances $\rho'(f_m(x), f(x))$, $x \in B$. Hence (2) follows.

Conversely, if

$$(\forall\, x \in B) \quad \rho'(f_m(x), f(x)) < \varepsilon,$$

then

$$\varepsilon \geq \sup_{x\in B} \rho'(f_m(x), f(x)),$$

i.e., $Q_m \leq \varepsilon$. Thus (2) implies

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, m > k) \quad Q_m \leq \varepsilon$$

and $Q_m \to 0$. $\square$

Examples.

(a) We have

$$\lim_{n\to\infty} x^n = 0 \text{ if } |x| < 1 \text{ and } \lim_{n\to\infty} x^n = 1 \text{ if } x = 1.$$

Thus, setting $f_n(x) = x^n$, consider $B = [0, 1]$ and $C = [0, 1)$.

We have $f_n \to 0$ (pointwise) on C and $f_n \to f$ (pointwise) on B, with $f(x) = 0$ for $x \in C$ and $f(1) = 1$. However, the limit *is not uniform on*

C, let alone on B. Indeed,

$$Q_n = \sup_{x \in C} |f_n(x) - f(x)| = 1 \text{ for each } n.^2$$

Thus Q_n does not tend to 0, and uniform convergence fails by Theorem 1.

(b) In Example (a), let $D = [0, a]$, $0 < a < 1$. Then $f_n \to f$ *(uniformly) on D* because, in this case,

$$Q_n = \sup_{x \in D} |f_n(x) - f(x)| = \sup_{x \in D} |x^n - 0| = a^n \to 0.$$

(c) Let

$$f_n(x) = x^2 + \frac{\sin nx}{n}, \quad x \in E^1.$$

For a fixed x,

$$\lim_{n \to \infty} f_n(x) = x^2 \quad \text{since} \left|\frac{\sin nx}{n}\right| \le \frac{1}{n} \to 0.$$

Thus, setting $f(x) = x^2$, we have $f_n \to f$ (pointwise) on E^1. Also,

$$|f_n(x) - f(x)| = \left|\frac{\sin nx}{n}\right| \le \frac{1}{n}.$$

Thus $(\forall n)\ Q_n \le \frac{1}{n} \to 0$. By Theorem 1, the limit is uniform on all of E^1.

Note 1. Example (a) shows that the *pointwise* limit of a sequence of continuous functions *need not* be continuous. Not so for *uniform* limits, as the following theorem shows.

Theorem 2. *Let $f_m : A \to (T, \rho')$ be a sequence of functions on $A \subseteq (S, \rho)$. If $f_m \to f$ (uniformly) on a set $B \subseteq A$, and if the f_m are relatively (or uniformly) continuous on B, then the limit function f has the same property.*

Proof. Fix $\varepsilon > 0$. As $f_m \to f$ (uniformly) on B, there is a k such that

$$(\forall x \in B)\ (\forall m \ge k) \quad \rho'(f_m(x), f(x)) < \frac{\varepsilon}{4}. \tag{3}$$

Take any f_m with $m > k$, and take any $p \in B$. By continuity, there is $\delta > 0$, with

$$(\forall x \in B \cap G_p(\delta)) \quad \rho'(f_m(x), f_m(p)) < \frac{\varepsilon}{4}. \tag{4}$$

[2] Here

$$Q_n = \sup_{x \in C} |x^n - 0| = \sup_{0 \le x < 1} x^n = \lim_{x \to 1} x^n = 1$$

by Theorem 1 of §5, because x^n increases with $x \nearrow 1$, i.e., each f_n is a monotone function on C. Note that *all f_n are continuous on $B = [0, 1]$*, but $f = \lim f_n$ is *discontinuous at* 1.

Also, setting $x = p$ in (3) gives $\rho'(f_m(p), f(p)) < \frac{\varepsilon}{4}$. Combining this with (4) and (3), we obtain $(\forall x \in B \cap G_p(\delta))$

$$\begin{aligned}\rho'(f(x), f(p)) &\le \rho'(f(x), f_m(x)) + \rho'(f_m(x), f_m(p)) + \rho'(f_m(p), f(p)) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.\end{aligned}$$

We thus see that for $p \in B$,

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x \in B \cap G_p(\delta)) \quad \rho'(f(x), f(p)) < \varepsilon,$$

i.e., f is *relatively* continuous at p (over B), as claimed.

Quite similarly, the reader will show that f is *uniformly* continuous if the f_n are. □

Note 2. A similar proof also shows that if $f_m \to f$ (uniformly) on B, and if the f_m are relatively continuous at *a point* $p \in B$, so also is f.

Theorem 3 (Cauchy criterion for uniform convergence)**.** *Let* (T, ρ') *be complete. Then a sequence* $f_m\colon A \to T$, $A \subseteq (S, \rho)$, *converges uniformly on a set* $B \subseteq A$ *iff*

$$(\forall \varepsilon > 0)\ (\exists k)\ (\forall x \in B)\ (\forall m, n > k) \quad \rho'(f_m(x), f_n(x)) < \varepsilon. \tag{5}$$

Proof. If (5) holds then, for any (fixed) $x \in B$, $\{f_m(x)\}$ *is a Cauchy sequence* of points in T, so by the assumed completeness of T, it has a limit $f(x)$. Thus we can define a function $f\colon B \to T$ with

$$f(x) = \lim_{m\to\infty} f_m(x) \text{ on } B.$$

To show that $f_m \to f$ (*uniformly*) on B, we use (5) again. Keeping ε, k, x, and m temporarily fixed, we let $n \to \infty$ so that $f_n(x) \to f(x)$. Then by Theorem 4 of Chapter 3, §15, $\rho'(f_m(x), f_n(x)) \to p'(f(x), f_m(x))$. Passing to the limit in (5), we thus obtain (2).

The easy proof of the converse is left to the reader (cf. Chapter 3, §17, Theorem 1). □

IV. If the range space (T, ρ') is E^1, C, or E^n (*or another normed space), the *standard* metric applies. In particular, for *series* we have

$$\begin{aligned}\rho'(s_m(x), s_n(x)) &= |s_n(x) - s_m(x)| \\ &= \left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)\right| \\ &= \left|\sum_{k=m+1}^{n} f_k(x)\right| \quad \text{for } m < n.\end{aligned}$$

Replacing here m by $m-1$ and applying Theorem 3 to the sequence $\{s_m\}$, we obtain the following result.

Theorem 3′. *Let the range space of* f_m, $m = 1, 2, \ldots,$ *be* E^1, C, *or* E^n (*or another* complete *normed space*). *Then the series* $\sum f_m$ *converges uniformly on* B *iff*

$$(\forall\, \varepsilon > 0)\ (\exists\, q)\ (\forall\, n > m > q)\ (\forall\, x \in B)\quad \left|\sum_{k=m}^{n} f_k(x)\right| < \varepsilon. \tag{6}$$

Similarly, via $\{s_m\}$, Theorem 2 extends to *series* of functions. (Observe that the s_m are continuous if the f_m are.) Formulate it!

V. If $\sum_{m=1}^{\infty} f_m$ *exists* on B, one may arbitrarily "group" the terms, i.e., replace every several consecutive terms by their sum. This property is stated more precisely in the following theorem.

Theorem 4. *Let*

$$f = \sum_{m=1}^{\infty} f_m \ (\textit{pointwise}) \ \textit{on } B.^3$$

Let $m_1 < m_2 < \cdots < m_n < \cdots$ *in* N, *and define*

$$g_1 = s_{m_1},\ g_n = s_{m_n} - s_{m_{n-1}}, \quad n > 1.$$

(*Thus* $g_{n+1} = f_{m_n+1} + \cdots + f_{m_{n+1}}$.) *Then*

$$f = \sum_{n=1}^{\infty} g_n \ (\textit{pointwise}) \ \textit{on } B \ \textit{as well};$$

similarly for uniform convergence.

Proof. Let

$$s'_n = \sum_{k=1}^{n} g_k, \quad n = 1, 2, \ldots.$$

Then $s'_n = s_{m_n}$ (verify!), so $\{s'_n\}$ is a *subsequence*, $\{s_{m_n}\}$, of $\{s_m\}$. Hence $s_m \to f$ (pointwise) *implies* $s'_n \to f$ (pointwise); i.e.,

$$f = \sum_{n=1}^{\infty} g_n \ (\text{pointwise}).$$

For uniform convergence, see Problem 13 (cf. also Problem 19). □

[3] Here we allow also *infinite* values for $f(x)$ if the f_m are *real*.

Problems on Sequences and Series of Functions

1. Complete the proof of Theorems 2 and 3.

2. Complete the proof of Theorem 4.

2′. In Example (a), show that $f_n \to +\infty$ (pointwise) on $(1, +\infty)$, but not uniformly so. Prove, however, that the limit is uniform on any interval $[a, +\infty)$, $a > 1$. (Define "$\lim f_n = +\infty$ (uniformly)" in a suitable manner.)

3. Using Theorem 1, discuss $\lim_{n\to\infty} f_n$ on B and C (as in Example (a)) for each of the following.

(i) $f_n(x) = \dfrac{x}{n}$; $B = E^1$; $C = [a, b] \subset E^1$.

(ii) $f_n(x) = \dfrac{\cos x + nx}{n}$; $B = E^1$.

(iii) $f_n(x) = \displaystyle\sum_{k=1}^{n} x^k$; $B = (-1, 1)$; $C = [-a, a]$, $|a| < 1$.

(iv) $f_n(x) = \dfrac{x}{1+nx}$; $C = [0, +\infty)$.

[Hint: Prove that $Q_n = \sup \frac{1}{n}\left(1 - \frac{1}{nx+1}\right) = \frac{1}{n}$.]

(v) $f_n(x) = \cos^n x$; $B = \left(0, \dfrac{\pi}{2}\right)$, $C = \left[\dfrac{1}{4}, \dfrac{\pi}{2}\right)$;

(vi) $f_n(x) = \dfrac{\sin^2 nx}{1+nx}$; $B = E^1$.

(vii) $f_n(x) = \dfrac{1}{1+x^n}$; $B = [0, 1)$; $C = [0, a]$, $0 < a < 1$.

4. Using Theorems 1 *and* 2, discuss $\lim f_n$ on the sets given below, with $f_n(x)$ as indicated and $0 < a < +\infty$. (Calculus rules for maxima and minima are assumed known in (v), (vi), and (vii).)

(i) $\dfrac{nx}{1+nx}$; $[a, +\infty)$, $(0, a)$.

(ii) $\dfrac{nx}{1+n^3x^3}$; $(a, +\infty)$, $(0, a)$.

(iii) $\sqrt[n]{\cos x}$; $\left(0, \dfrac{\pi}{2}\right)$, $[0, a]$, $a < \dfrac{\pi}{2}$.

(iv) $\dfrac{x}{n}$; $(0, a)$, $(0, +\infty)$.

(v) xe^{-nx}; $[0, +\infty)$; E^1.

(vi) nxe^{-nx}; $[a, +\infty)$, $(0, +\infty)$.

(vii) nxe^{-nx^2}; $[a, +\infty)$, $(0, +\infty)$.

[Hint: $\lim f_n$ cannot be uniform if the f_n are continuous on a set, but $\lim f_n$ is not. For (v), f_n has a maximum at $x = \frac{1}{n}$; hence find Q_n.]

5. Define $f_n\colon E^1 \to E^1$ by

$$f_n(x) = \begin{cases} nx & \text{if } 0 \le x \le \frac{1}{n}, \\ 2 - nx & \text{if } \frac{1}{n} < x \le \frac{2}{n}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Show that all f_n and $\lim f_n$ are continuous on each interval $(-a, a)$, though $\lim f_n$ exists only pointwise. (Compare this with Theorem 3.)

6. The function f found in the proof of Theorem 3 is *uniquely* determined. Why?

⇒7. Prove that if the functions f_n are constant on B, or if B is finite, then a pointwise limit of the f_n on B is also uniform; similarly for series.

⇒8. Prove that if $f_n \to f$ (uniformly) on B and if $C \subseteq B$, then $f_n \to f$ (uniformly) on C as well.

⇒9. Show that if $f_n \to f$ (uniformly) on each of $B_1, B_2, \ldots, B_m$, then $f_n \to f$ (uniformly) on $\bigcup_{k=1}^{m} B_k$.

Disprove it for infinite unions by an example. Do the same for series.

⇒10. Let $f_n \to f$ (uniformly) on B. Prove the equivalence of the following statements:

(i) Each f_n, from a certain n onward, is bounded on B.

(ii) f is bounded on B.

(iii) The f_n are ultimately *uniformly bounded on* B; that is, all function values $f_n(x)$, $x \in B$, from a certain $n = n_0$ onward, are in one and the same globe $G_q(K)$ in the range space.

For real, complex, and vector-valued functions, this means that

$$(\exists\, K \in E^1)\ (\forall\, n \ge n_0)\ (\forall\, x \in B) \quad |f_n(x)| < K.$$

⇒11. Prove for real, complex, or vector-valued functions f_n, f, g_n, g that if

$$f_n \to f \text{ and } g_n \to g \text{ (uniformly) on } B,$$

then also

$$f_n \pm g_n \to f \pm g \text{ (uniformly) on } B.$$

⇒12. Prove that if the functions f_n and g_n are real or complex (or if the g_n are vector valued and the f_n are scalar valued), and if

$$f_n \to f \text{ and } g_n \to g \text{ (uniformly) on } B,$$

then

$$f_n g_n \to fg \text{ (uniformly) on } B$$

provided that *either* f *and* g *or the* f_n *and* g_n *are bounded on* B (at least from some n onward); cf. Problem 11.

Disprove it for the case where only one of f and g is bounded.

[Hint: Let $f_n(x) = x$ and $g_n(x) = 1/n$ (constant) on $B = E^1$. Give some other examples.]

⇒**13.** Prove that if $\{f_n\}$ tends to f (pointwise or uniformly), so does each subsequence $\{f_{n_k}\}$.

⇒**14.** Let the functions f_n and g_n and the constants a and b be real or complex (or let a and b be scalars and f_n and g_n be vector valued). Prove that if

$$f = \sum_{n=1}^{\infty} f_n \text{ and } g = \sum_{n=1}^{\infty} g_n \text{ (pointwise or uniformly),}$$

then

$$af + bg = \sum_{n=1}^{\infty} (af_n + bg_n) \text{ in the same sense.}$$

(Infinite limits are excluded.)

In particular,

$$f \pm g = \sum_{n=1}^{\infty} (f_n \pm g_n) \quad \text{(rule of termwise addition)}$$

and

$$af = \sum_{n=1}^{\infty} af_n.$$

[Hint: Use Problems 11 and 12.]

⇒**15.** Let the range space of the functions f_m and g be E^n (*or C^n), and let $f_m = (f_{m1}, f_{m2}, \ldots, f_{mn})$, $g = (g_1, \ldots, g_n)$; see §3, part II. Prove that

$$f_m \to g \quad \text{(pointwise or uniformly)}$$

iff each component f_{mk} of f_m converges (in the same sense) to the corresponding component g_k of g; i.e.,

$$f_{mk} \to g_k \quad \text{(pointwise or uniformly), } k = 1, 2, \ldots, n.$$

Similarly,

$$g = \sum_{m=1}^{\infty} f_m$$

iff

$$(\forall\, k \le n) \quad g_k = \sum_{m=1}^{\infty} f_{mk}.$$

(See Chapter 3, §15, Theorem 2).

⇒**16.** From Problem 15 deduce for *complex* functions that $f_m \to g$ (pointwise or uniformly) iff the real and imaginary parts of the f_m converge to those of g (pointwise or uniformly). That is, $(f_m)_{re} \to g_{re}$ and $(f_m)_{im} \to g_{im}$; similarly for series.

⇒**17.** Prove that the convergence or divergence (pointwise or uniformly) of a sequence $\{f_m\}$, or a series $\sum f_m$, of functions is not affected by deleting or adding a finite number of terms.

Prove also that $\lim_{m\to\infty} f_m$ (if any) remains the same, but $\sum_{m=1}^{\infty} f_m$ is altered by the difference between the added and deleted terms.

⇒**18.** Show that the *geometric series with ratio* r,

$$\sum_{n=0}^{\infty} ar^n \quad (a, r \in E^1 \text{ or } a, r \in C),$$

converges iff $|r| < 1$, in which case

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

(similarly if a is a vector and r is a scalar). Deduce that $\sum(-1)^n$ diverges. (See Chapter 3, §15, Problem 19.)

19. Theorem 4 shows that a convergent series does not change its sum if every several consecutive terms are replaced by their sum. Show by an example that the *reverse* process (splitting each term into several terms) may affect convergence.

[Hint: Consider $\sum a_n$ with $a_n = 0$. Split $a_n = 1 - 1$ to obtain a *divergent* series: $\sum(-1)^{n-1}$, with partial sums $1, 0, 1, 0, 1, \ldots$.]

20. Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

[Hint: Verify: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Hence find s_n, and let $n \to \infty$.]

21. The functions $f_n \colon A \to (T, \rho')$, $A \subseteq (S, \rho)$ are said to be *equicontinuous* at $p \in A$ iff

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, n)\ (\forall\, x \in A \cap G_p(\delta)) \quad \rho'(f_n(x), f_n(p)) < \varepsilon.$$

Prove that if so, and if $f_n \to f$ (pointwise) on A, then f is continuous at p.

[Hint: "Imitate" the proof of Theorem 2.]

§13. Absolutely Convergent Series. Power Series

I. A series $\sum f_m$ is said to be *absolutely convergent* on a set B iff the series $\sum |f_m(x)|$ (briefly, $\sum |f_m|$) of the *absolute values* of f_m converges on B (pointwise or uniformly). Notation:

$$f = \sum |f_m| \text{ (pointwise or uniformly) on } B.$$

In general, $\sum f_m$ may converge while $\sum |f_m|$ does not (see Problem 12). In this case, the convergence of $\sum f_m$ is said to be *conditional*. (It may be absolute for some x and conditional for others.) As we shall see, absolute convergence ensures the *commutative law* for series, and *it implies ordinary convergence* (i.e., that of $\sum f_m$), if the range space of the f_m is *complete*.

Note 1. Let

$$\sigma_m = \sum_{k=1}^{m} |f_k|.$$

Then

$$\sigma_{m+1} = \sigma_m + |f_{m+1}| \geq \sigma_m \quad \text{on } B;[1]$$

i.e., the $\sigma_m(x)$ form a *monotone* sequence for each $x \in B$. Hence by Theorem 3 of Chapter 3, §15,

$$\lim_{m\to\infty} \sigma_m = \sum_{m=1}^{\infty} |f_m| \textit{ always } \text{exists } \textit{in } E^*;$$

$\sum |f_m|$ *converges iff* $\sum_{m=1}^{\infty} |f_m| < +\infty$.

For the rest of this section we consider only *complete* range spaces.

Theorem 1. *Let the range space of the functions* f_m (*all defined on* A) *be* E^1, C, *or* E^n (**or another* complete *normed space*). *Then for* $B \subseteq A$, *we have the following*:

(i) *If* $\sum |f_m|$ *converges on* B (*pointwise or uniformly*), *so does* $\sum f_m$ *itself. Moreover*,

$$\left|\sum_{m=1}^{\infty} f_m\right| \leq \sum_{m=1}^{\infty} |f_m| \quad \textit{on } B.$$

(ii) (*Commutative law for absolute convergence.*) *If* $\sum |f_m|$ *converges* (*pointwise or uniformly*) *on* B, *so does any series* $\sum |g_m|$ *obtained by rearrang-*

[1] We write "$f \leq g$ on B" for "$(\forall\, x \in B)\ f(x) \leq g(x)$;" similarly for such formulas as "$f = g$ on B," "$|f| < +\infty$ on B," "$f = c$ (constant) on B," etc.

ing the f_m *in any different order.*[2] *Moreover,*

$$\sum_{m=1}^{\infty} f_m = \sum_{m=1}^{\infty} g_m \quad (\textit{both exist on } B).$$

Note 2. More precisely, a sequence $\{g_m\}$ is called a *rearrangement* of $\{f_m\}$ iff there is a map $u\colon N \underset{\text{onto}}{\longleftrightarrow} N$ such that

$$(\forall\, m \in N) \quad g_m = f_{u(m)}.$$

Proof.

(i) If $\sum |f_m|$ converges uniformly on B, then by Theorem 3′ of §12,

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, n > m > k)\ (\forall\, x \in B)$$
$$\varepsilon > \sum_{k=m}^{n} |f_k(x)| \geq \left|\sum_{k=m}^{n} f_k(x)\right| \quad \text{(triangle law).} \tag{1}$$

However, this shows that $\sum f_n$ satisfies Cauchy's criterion (6) of §12, so it converges uniformly on B.

Moreover, letting $n \to \infty$ in the inequality

$$\left|\sum_{m=1}^{n} f_m\right| \leq \sum_{m=1}^{n} |f_m|,$$

we get

$$\left|\sum_{m=1}^{\infty} f_m\right| \leq \sum_{m=1}^{\infty} |f_m| < +\infty \quad \text{on } B \text{, as claimed.}$$

By Note 1, this also proves the theorem for *pointwise* convergence.

(ii) Again, if $\sum |f_m|$ converges uniformly on B, the inequalities (1) hold for all f_i except (possibly) for $f_1, f_2, \ldots, f_k$. Now when $\sum f_m$ is *rearranged*, these k functions will be renumbered as certain g_i. Let q be the largest of their *new* subscripts i. Then all of them (and possibly some more functions) are among $g_1, g_2, \ldots, g_q$ (so that $q \geq k$). Hence if we exclude $g_1, \ldots, g_q$, the inequalities (1) will certainly hold for the remaining g_i $(i > q)$. Thus

$$(\forall\, \varepsilon > 0)\ (\exists\, q)\ (\forall\, n > m > q)\ (\forall\, x \in B) \quad \varepsilon > \sum_{i=m}^{n} |g_i| \geq \left|\sum_{i=m}^{n} g_i\right|. \tag{2}$$

By Cauchy's criterion, then, both $\sum g_i$ and $\sum |g_i|$ converge uniformly.

[2] This fails for *conditional* convergence. See Problem 17.

Moreover, by construction, the two partial sums

$$s_k = \sum_{i=1}^{k} f_i \text{ and } s'_q = \sum_{i=1}^{q} g_i$$

can differ only in those terms whose *original* subscripts (before the rearrangement) were $> k$. By (1), however, *any finite sum of such terms is less than ε in absolute value*. Thus $|s'_q - s_k| < \varepsilon$.

This argument holds also if k in (1) is replaced by a larger integer. (Then also q increases, since $q \geq k$ as noted above.) Thus we may let $k \to +\infty$ (hence also $q \to +\infty$) in the inequality $|s'_q - s_k| < \varepsilon$, with ε fixed. Then

$$s_k \to \sum_{m=1}^{\infty} f_m \text{ and } s'_q \to \sum_{i=1}^{\infty} g_i,$$

so

$$\left| \sum_{i=1}^{\infty} g_i - \sum_{m=1}^{\infty} f_m \right| \leq \varepsilon.$$

Now let $\varepsilon \to 0$ to get

$$\sum_{i=1}^{\infty} g_i = \sum_{m=1}^{\infty} f_m;$$

similarly for pointwise convergence. □

II. Next, we develop some simple tests for *absolute* convergence.

Theorem 2 (comparison test)**.** *Suppose*

$$(\forall\, m) \quad |f_m| \leq |g_m| \text{ on } B.$$

Then

(i) $\displaystyle\sum_{m=1}^{\infty} |f_m| \leq \sum_{m=1}^{\infty} |g_m|$ *on* B;

(ii) $\displaystyle\sum_{m=1}^{\infty} |f_m| = +\infty$ *implies* $\displaystyle\sum_{m=1}^{\infty} |g_m| = +\infty$ *on* B; *and*

(iii) *If* $\sum |g_m|$ *converges (pointwise or uniformly) on* B, *so does* $\sum |f_m|$.

Proof. Conclusion (i) follows by letting $n \to \infty$ in

$$\sum_{m=1}^{n} |f_m| \leq \sum_{m=1}^{n} |g_m|.$$

In turn, (ii) is a direct consequence of (i).

Also, by (i),

$$\sum_{m=1}^{\infty} |g_m| < +\infty \text{ implies } \sum_{m=1}^{\infty} |f_m| < +\infty.$$

This proves (iii) for the *pointwise* case (see Note 1). The uniform case follows exactly as in Theorem 1(i) on noting that

$$\sum_{k=m}^{n} |f_k| \leq \sum_{k=m}^{n} |g_k|$$

and that the functions $|f_k|$ and $|g_k|$ are *real* (so Theorem 3′ in §12 *does* apply). □

Theorem 3 (Weierstrass "M-test")**.** *If $\sum M_n$ is a convergent series of real constants $M_n \geq 0$ and if*

$$(\forall\, n) \quad |f_n| \leq M_n$$

on a set B, then $\sum |f_n|$ converges uniformly on B.[3] *Moreover,*

$$\sum_{n=1}^{\infty} |f_n| \leq \sum_{n=1}^{\infty} M_n \quad \textit{on } B.$$

Proof. Use Theorem 2 with $|g_n| = M_n$, noting that $\sum |g_n|$ converges *uniformly* since the $|g_n|$ are *constant* (§12, Problem 7). □

Examples.

(a) Let

$$f_n(x) = \left(\frac{1}{2}\sin x\right)^n \text{ on } E^1.$$

Then

$$(\forall\, n)\ (\forall\, x \in E^1) \quad |f_n(x)| \leq 2^{-n},$$

and $\sum 2^{-n}$ converges (geometric series with ratio $\frac{1}{2}$; see §12, Problem 18). Thus, setting $M_n = 2^{-n}$ in Theorem 3, we infer that the series $\sum |\frac{1}{2}\sin x|^n$ converges *uniformly* on E^1, as does $\sum(\frac{1}{2}\sin x)^n$; moreover,

$$\sum_{n=1}^{\infty} |f_n| \leq \sum_{n=1}^{\infty} 2^{-n} = 1.$$

[3] So does $\sum f_n$ itself if the range space is as in Theorem 1. Note that for series with positive terms, absolute and ordinary convergence coincide.

Theorem 4 (necessary condition of convergence)**.** *If* $\sum f_m$ *or* $\sum |f_m|$ *converges on* B *(pointwise or uniformly), then* $|f_m| \to 0$ *on* B *(in the same sense).*

Thus a series *cannot* converge unless its general term tends to 0 (respectively, $\bar{0}$).

Proof. If $\sum f_m = f$, say, then $s_m \to f$ and also $s_{m-1} \to f$. Hence

$$s_m - s_{m-1} \to f - f = \bar{0}.$$

However, $s_m - s_{m-1} = f_m$. Thus $f_m \to \bar{0}$, and $|f_m| \to 0$, as claimed.

This holds for pointwise and uniform convergence alike (see Problem 14 in §12). $\square$

Caution: The condition $|f_m| \to 0$ is necessary but *not sufficient*. Indeed, there are *divergent* series with general term tending to 0, as we show next.

Examples (continued)**.**

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$ (the so-called *harmonic* series).

Indeed, by Note 1,

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \textit{exists} \text{ (in } E^*\text{)},$$

so Theorem 4 of §12 applies. We group the series as follows:

$$\begin{aligned}\sum \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots \\ &\geq \frac{1}{2} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) + \cdots .\end{aligned}$$

Each bracketed expression now equals $\frac{1}{2}$. Thus

$$\sum \frac{1}{n} \geq \sum g_m, \quad g_m = \frac{1}{2}.$$

As g_m does not tend to 0, $\sum g_m$ *diverges*, i.e., $\sum_{m=1}^{\infty} g_m$ is infinite, by Theorem 4. A fortiori, so is $\sum_{n=1}^{\infty} \frac{1}{n}$.

Theorem 5 (root and ratio tests)**.** *A series of constants* $\sum a_n$ $(|a_n| \neq 0)$ *converges absolutely if*

$$\overline{\lim}\, \sqrt[n]{|a_n|} < 1 \ \textit{or}\ \overline{\lim}\left(\frac{|a_{n+1}|}{|a_n|}\right) < 1.$$

It diverges if

$$\overline{\lim}\ \sqrt[n]{|a_n|} > 1 \text{ or } \underline{\lim}\Big(\frac{|a_{n+1}|}{|a_n|}\Big) > 1.^{4}$$

It may converge or diverge if

$$\overline{\lim}\ \sqrt[n]{|a_n|} = 1$$

or if

$$\underline{\lim}\Big(\frac{|a_{n+1}|}{|a_n|}\Big) \le 1 \le \overline{\lim}\Big(\frac{|a_{n+1}|}{|a_n|}\Big).$$

(*The* a_n *may be scalars or vectors.*)

Proof. If $\overline{\lim}\ \sqrt[n]{|a_n|} < 1$, choose $r > 0$ such that

$$\overline{\lim}\ \sqrt[n]{|a_n|} < r < 1.$$

Then by Corollary 2 of Chapter 2, §13, $\sqrt[n]{|a_n|} < r$ for all but finitely many n. Thus, dropping a finite number of terms (§12, Problem 17), we may assume that

$$|a_n| < r^n \text{ for } all\ n.$$

As $0 < r < 1$, the geometric series $\sum r^n$ converges. Hence so does $\sum |a_n|$ by Theorem 2.

In the case

$$\overline{\lim}\Big(\frac{|a_{n+1}|}{|a_n|}\Big) < 1,$$

we similarly obtain $(\exists\, m)\ (\forall\, n \ge m)\ |a_{n+1}| < |a_n| r$; hence by induction,

$$(\forall\, n \ge m) \quad |a_n| \le |a_m| r^{n-m}. \quad \text{(Verify!)}$$

Thus $\sum |a_n|$ converges, as before.

If $\overline{\lim}\ \sqrt[n]{|a_n|} > 1$, then by Corollary 2 of Chapter 2, §13, $|a_n| > 1$ for *infinitely many* n. Hence $|a_n|$ *cannot tend to* 0, and so $\sum a_n$ diverges by Theorem 4.

Similarly, if

$$\underline{\lim}\Big(\frac{|a_{n+1}|}{|a_n|}\Big) > 1,$$

then $|a_{n+1}| > |a_n|$ for *all but finitely many* n, so $|a_n|$ cannot tend to 0 again.[5] □

Note 3. We have

$$\underline{\lim}\Big(\frac{|a_{n+1}|}{|a_n|}\Big) \le \underline{\lim}\ \sqrt[n]{|a_n|} \le \overline{\lim}\ \sqrt[n]{|a_n|} \le \overline{\lim}\Big(\frac{|a_{n+1}|}{|a_n|}\Big).^{6}$$

---

[4] Note that we have "$\underline{\lim}$", not "$\overline{\lim}$" here. However, often "$\underline{\lim}$" and "$\overline{\lim}$" coincide. This is the case when the *limit* exists (see Chapter 2, §13, Theorem 3).

[5] This inference would be false if we only had $\overline{\lim}(|a_{n+1}|/|a_n|) > 1$. Why?

[6] For a proof, use Problem 33 of Chapter 3, §15 with $x_1 = |a_1|$ and $x_{k+1} = |a_{k+1}|/|a_k|$.

Thus

$$\overline{\lim}\Big(\frac{|a_{n+1}|}{|a_n|}\Big) < 1 \textit{ implies } \overline{\lim}\ \sqrt[n]{|a_n|} < 1; \text{ and}$$

$$\underline{\lim}\Big(\frac{|a_{n+1}|}{|a_n|}\Big) > 1 \textit{ implies } \overline{\lim}\ \sqrt[n]{|a_n|} > 1.$$

Hence *whenever the ratio test indicates convergence or divergence, so certainly does the root test.* On the other hand, there are cases where the root test yields a result while the ratio test does not. Thus the root test is stronger (but the ratio test is often easier to apply).

Examples (continued)**.**

(c) Let $a_n = 2^{-k}$ if $n = 2k-1$ (odd) and $a_n = 3^{-k}$ if $n = 2k$ (even). Thus

$$\sum a_n = \frac{1}{2^1} + \frac{1}{3^1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots .$$

Here

$$\underline{\lim}\Big(\frac{a_{n+1}}{a_n}\Big) = \lim_{k\to\infty} \frac{3^{-k}}{2^{-k}} = 0 \text{ and } \overline{\lim}\Big(\frac{a_{n+1}}{a_n}\Big) = \lim_{k\to\infty} \frac{2^{-k-1}}{3^{-k}} = +\infty,$$

while

$$\overline{\lim}\ \sqrt[n]{a_n} = \lim\ \sqrt[2n-1]{2^{-n}} = \frac{1}{\sqrt{2}} < 1.^{7} \quad \text{(Verify!)}$$

Thus the ratio test fails, but the root test proves convergence.

Note 4. The assumption $|a_n| \neq 0$ is needed for the ratio test only.

III. Power Series. As an application, we now consider so-called *power series*,

$$\sum a_n (x-p)^n,$$

where x, p, $a_n \in E^1$ (C); the a_n may also be *vectors*.

Theorem 6. *For any power series* $\sum a_n(x-p)^n$, *there is a unique* $r \in E^*$ $(0 \leq r \leq +\infty)$, *called its* convergence radius, *such that the series converges absolutely for each* x *with* $|x-p| < r$ *and does not converge* (*even conditionally*) *if* $|x-p| > r$.[8]

Specifically,

$$r = \frac{1}{d}, \textit{ where } d = \overline{\lim}\ \sqrt[n]{|a_n|} \quad (\textit{with } r = +\infty \textit{ if } d = 0).$$

[7] Recall that $\underline{\lim}$ and $\overline{\lim}$ are *cluster points*, hence limits of suitable subsequences. See Chapter 2, §13, Problem 4 and Chapter 3, §16, Theorem 1.

[8] The case $|x-p| = r$ remains open.

Proof. Fix any $x = x_0$. By Theorem 5, the series $\sum a_n(x_0 - p)^n$ converges absolutely if $\overline{\lim} \sqrt[n]{|a_n|}\, |x_0 - p| < 1$, i.e., if

$$|x_0 - p| < r \quad \left(r = \frac{1}{\overline{\lim} \sqrt[n]{|a|}} = \frac{1}{d}\right),$$

and diverges if $|x_0 - p| > r$. (Here we assumed $d \neq 0$; but if $d = 0$, the condition $d|x_0 - p| < 1$ is trivial for *any* x_0, so $r = +\infty$ in this case.) Thus r is the required radius, and clearly there can be only *one* such r. (Why?) $\square$

Note 5. If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists, it *equals* $\lim_{n\to\infty} \sqrt[n]{|a_n|}$, by Note 3 (for $\overline{\lim}$ and $\underline{\lim}$ coincide here). In this case, one can use the *ratio test* to find

$$d = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

and hence (if $d \neq 0$)

$$r = \frac{1}{d} = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}.$$

Theorem 7. *If a power series* $\sum a_n(x - p)^n$ *converges absolutely for some* $x = x_0 \neq p$, *then* $\sum |a_n(x-p)^n|$ *converges uniformly on the closed globe* $\overline{G}_p(\delta)$, $\delta = |x_0 - p|$. *So does* $\sum a_n(x-p)^n$ *if the range space is complete* (*Theorem* 1).

Proof. Suppose $\sum |a_n(x_0 - p)^n|$ converges. Let

$$\delta = |x_0 - p| \text{ and } M_n = |a_n|\delta^n;$$

thus $\sum M_n$ converges.

Now if $x \in \overline{G}_p(\delta)$, then $|x - p| \leq \delta$, so

$$|a_n(x - p)^n| \leq |a_n|\delta^n = M_n.$$

Hence by Theorem 3, $\sum |a_n(x - p)^n|$ converges uniformly on $\overline{G}_p(\delta)$. $\square$

Examples (continued)**.**

(d) Consider $\sum \frac{x^n}{n!}$ Here

$$p = 0 \text{ and } a_n = \frac{1}{n!}, \text{ so } \frac{|a_n|}{|a_{n+1}|} = n + 1 \to +\infty.$$

By Note 5, then, $r = +\infty$; i.e., the series converges absolutely on all of E^1. Hence by Theorem 7, it converges uniformly on *any* $\overline{G}_0(\delta)$, hence on any finite interval in E^1. (The *pointwise* convergence is on all of E^1.)

More Problems on Series of Functions

1. Verify Note 3 and Example (c) in detail.

2. Show that the so-called *hyperharmonic series of order p*,

$$\sum \frac{1}{n^p} \quad (p \in E^1),$$

converges iff $p > 1$.

[Hint: If $p \le 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \ge \sum_{n=1}^{\infty} \frac{1}{n} = +\infty \quad \text{(Example (b))}.$$

If $p > 1$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^p} &= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \cdots + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \cdots + \frac{1}{15^p}\right) + \cdots \\ &\le 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \cdots + \frac{1}{4^p}\right) + \left(\frac{1}{8^p} + \cdots + \frac{1}{8^p}\right) + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{(2^{p-1})^n},\end{aligned}$$

a convergent geometric series. Explain each step.]

⇒**3.** Prove the *refined comparison test*:

(i) If two series of constants, $\sum |a_n|$ and $\sum |b_n|$, are such that the sequence $\{|a_n|/|b_n|\}$ is *bounded* in E^1, then

$$\sum_{n=1}^{\infty} |b_n| < +\infty \text{ implies } \sum_{n=1}^{\infty} |a_n| < +\infty.$$

(ii) If

$$0 < \lim_{n\to\infty} \frac{|a_n|}{|b_n|} < +\infty,$$

then $\sum |a_n|$ converges *if and only if* $\sum |b_n|$ does.

What if

$$\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = +\infty?$$

[Hint: If $(\forall\, n)$ $|a_n|/|b_n| \le K$, then $|a_n| \le K|b_n|$.]

4. Test $\sum a_n$ for absolute convergence in each of the following. Use Problem 3 or Theorem 2 or the indicated references.

(i) $a_n = \dfrac{n+1}{\sqrt{n^4+1}}$ (take $b_n = \dfrac{1}{n}$);

(ii) $a_n = \dfrac{\cos n}{\sqrt{n^3-1}}$ (take $b_n = \dfrac{1}{\sqrt{n^3}}$; use Problem 2);

(iii) $a_n = \dfrac{(-1)^n}{n^p}(\sqrt{n+1} - \sqrt{n})$;

(iv) $a_n = n^5 e^{-n}$ (use Problem 18 of Chapter 3, §15);

(v) $a_n = \dfrac{2^n + n}{3^n + 1}$;

(vi) $a_n = \dfrac{(-1)^n}{(\log n)^q}; n \geq 2$;

(vii) $a_n = \dfrac{(\log n)^q}{n(n^2+1)}$, $q \in E^1$.

[Hint for (vi) and (vii): From Problem 14 in §2, show that

$$\lim_{y \to +\infty} \frac{y}{(\log y)^q} = +\infty$$

and hence

$$\lim_{n \to \infty} \frac{(\log n)^q}{n} = 0.$$

Then select b_n.]

5. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!} = +\infty$.

[Hint: Show that $n^n/n!$ does not tend to 0.]

6. Prove that $\displaystyle\lim_{n \to \infty} \frac{x^n}{n!} = 0$.

[Hint: Use Example (d) and Theorem 4.]

7. Use Theorems 3, 5, 6, and 7 to show that $\sum |f_n|$ converges uniformly on B, provided $f_n(x)$ and B are as indicated below, with $0 < a < +\infty$ and $b \in E^1$.[9] For parts (ix)–(xii), find $M_n = \max_{x \in B} |f_n(x)|$ and use Theorem 3. (Calculus rules for maxima are assumed known.)

(i) $\dfrac{x^{2n}}{(2n)!}$; $[-a, b]$.

(ii) $(-1)^{n+1} \dfrac{x^{2n-1}}{(2n-1)!}$; $[-a, b]$.

(iii) $\dfrac{x^n}{n^n}$; $[-a, a]$.

(iv) $n^3 x^n$; $[-a, a]$ $(a < 1)$.

(v) $\dfrac{\sin nx}{n^2}$; $B = E^1$ (use Problem 2).

(vi) $e^{-nx} \sin nx$; $[a, +\infty)$.

(vii) $\dfrac{\cos nx}{\sqrt{n^3+1}}$; $B = E^1$.

[9] For power series, do it in *two* ways and find the radius of convergence.

(viii) $a_n \cos nx$, with $\sum_{n=1}^{\infty} |a_n| < +\infty$; $B = E^1$.

(ix) $x^n e^{-nx}$; $[0, +\infty)$.

(x) $x^n e^{nx}$; $(-\infty, \frac{1}{2}]$.

(xi) $(x \cdot \log x)^n$, $f_n(0) = 0$; $\left[-\frac{3}{2}, \frac{3}{2}\right]$.

(xii) $\left(\frac{\log x}{x}\right)^n$; $[1, +\infty)$.

(xiii) $\frac{q(q-1)\cdots(q-n+1)x^n}{n!}$, $q \in E^1$; $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

⇒**8.** (*Summation by parts.*) Let f_n, h_n, and g_n be real or complex functions (or let f_n and h_n be scalar valued and g_n be vector valued). Let $f_n = h_n - h_{n-1}$ $(n \geq 2)$. Verify that $(\forall\, m > n > 1)$

$$\sum_{k=n+1}^{m} f_k g_k = \sum_{k=n+1}^{m} (h_k - h_{k-1}) g_k$$
$$= h_m g_m - h_n g_{n+1} - \sum_{k=n+1}^{m-1} h_k (g_{k+1} - g_k).$$

[Hint: Rearrange the sum.]

⇒**9.** (*Abel's test.*) Let the f_n, g_n, and h_n be as in Problem 8, with $h_n = \sum_{i=1}^{n} f_i$. Suppose that

(i) the range space of the g_n is complete;

(ii) $|g_n| \to 0$ (uniformly) on a set B; and

(iii) the partial sums $h_n = \sum_{i=1}^{n} f_i$ are *uniformly bounded* on B; i.e.,

$$(\exists\, K \in E^1)\ (\forall\, n) \quad |h_n| < K \text{ on } B.$$

Then prove that $\sum f_k g_k$ converges uniformly on B if $\sum |g_{n+1} - g_n|$ does. (This *always* holds if the g_n are real and $g_n \geq g_{n+1}$ on B.)
[Hint: Let $\varepsilon > 0$. Show that

$$(\exists\, k)\ (\forall\, m > n > k) \quad \sum_{i=n+1}^{m} |g_{i+1} - g_i| < \varepsilon \text{ and } |g_n| < \varepsilon \text{ on } B.$$

Then use Problem 8 to show that

$$\left|\sum_{i=n+1}^{m} f_i g_i\right| < 3K\varepsilon.$$

Apply Theorem 3′ of §12.]

⇒ 9′. Prove that if $\sum a_n$ is a convergent series of constants $a_n \in E^1$ and if $\{b_n\}$ is a bounded monotone sequence in E^1, then $\sum a_n b_n$ converges.
[Hint: Let $b_n \to b$. Write

$$a_n b_n = a_n(b_n - b) + a_n b$$

and use Problem 9 with $f_n = a_n$ and $g_n = b_n - b$.]

⇒10. Prove the *Leibniz test for alternating series*: If $\{b_n\}\downarrow$ and $b_n \to 0$ in E^1, then $\sum(-1)^n b_n$ converges, and the sum $\sum_{n=1}^{\infty}(-1)^n b_n$ differs from $s_n = \sum_{k=1}^{n}(-1)^k b_k$ by b_{n+1} at most.
[Hint: Use Problem 9′.]

⇒11. (*Dirichlet test.*) Let the f_n, g_n, and h_n be as in Problem 8 with $\sum_{n=0}^{\infty} f_n$ uniformly convergent on B to a function f, and with

$$h_n = -\sum_{i=n+1}^{\infty} f_i \text{ on } B.$$

Suppose that

(i) the range space of the g_n is complete; and

(ii) there is $K \in E^1$ such that

$$|g_0| + \sum_{n=0}^{\infty} |g_{n+1} - g_n| < K \text{ on } B.$$

Show that $\sum f_n g_n$ converges uniformly on B.
[Proof outline: We have

$$|g_n| = \left|g_0 + \sum_{i=0}^{n-1}(g_{i+1} - g_i)\right| \le |g_0| + \sum_{i=0}^{n-1}|g_{i+1} - g_i| < K \quad \text{by (ii)}.$$

Also,

$$|h_n| = \left|\sum_{i=0}^{n} f_i - f\right| \to 0 \text{ (uniformly) on } B$$

by assumption. Hence

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, n > k) \quad |h_n| < \varepsilon \text{ on } B.$$

Using Problem 8, obtain

$$(\forall\, m > n > k) \quad \left|\sum_{i=n+i}^{m} f_i g_i\right| < 2K\varepsilon.]$$

12. Prove that if $0 < p \le 1$, then $\sum \dfrac{(-1)^n}{n^p}$ converges *conditionally*.
[Hint: Use Problems 11 and 2.]

⇒13. Continuing Problem 14 in §12, prove that if $\sum|f_n|$ and $\sum|g_n|$ converge on B (pointwise or uniformly), then so do the series

$$\sum|af_n + bg_n|,\ \sum|f_n \pm g_n|, \text{ and } \sum|af_n|.$$

[Hint: $|af_n + bg_n| \le |a||f_n| + |b||g_n|$. Use Theorem 2.]

For the rest of the section, we define

$$x^+ = \max(x, 0) \text{ and } x^- = \max(-x,\, 0).$$

⇒14. Given $\{a_n\} \subset E^*$ show the following:

(i) $\sum a_n^+ + \sum a_n^- = \sum|a_n|$.

(ii) If $\sum a_n^+ < +\infty$ or $\sum a_n^- < +\infty$, then $\sum a_n = \sum a_n^+ - \sum a_n^-$.

(iii) If $\sum a_n$ converges *conditionally*, then $\sum a_n^+ = +\infty = \sum a_n^-$.

(iv) If $\sum|a_n| < +\infty$, then for any $\{b_n\} \subset E^1$,

$$\sum|a_n \pm b_n| < +\infty \text{ iff } \sum|b_n| < \infty;$$

moreover, $\sum a_n \pm \sum b_n = \sum(a_n \pm b_n)$ if $\sum b_n$ exists.

[Hint: Verify that $|a_n| = a_n^+ + a_n^-$ and $a_n = a_n^+ - a_n^-$. Use the rules of §4.]

⇒15. (*Abel's theorem.*) Show that if a power series

$$\sum_{n=0}^{\infty} a_n(x-p)^n \quad (a_n \in E,\, x,\, p \in E^1)$$

converges for some $x = x_0 \neq p$, it converges *uniformly on* $[p,\, x_0]$ (*or* $[x_0,\, p]$ if $x_0 < p$).

[Proof outline: First let $p = 0$ and $x_0 = 1$. Use Problem 11 with

$$f_n = a_n \text{ and } g_n(x) = x^n = (x-p)^n.$$

As $f_n = a_n 1^n = a_n(x_0 - p)^n$, the series $\sum f_n$ converges by assumption. The convergence is uniform since the f_n are *constant*. Verify that if $x = 1$, then

$$\sum_{k=1}^{\infty}|g_{k+1} - g_k| = 0,$$

and if $0 \le x < 1$, then

$$\sum_{k=0}^{\infty}|g_{k+1} - g_k| = \sum_{k=0}^{\infty} x^k|x-1| = (1-x)\sum_{k=0}^{\infty} x^k = 1 \quad \text{(a *geometric series*)}.$$

Also, $|g_0(x)| = x^0 = 1$. Thus by Problem 11 (with $K = 2$), $\sum f_n g_n$ converges uniformly on $[0,\, 1]$, proving the theorem for $p = 0$ and $x_0 = 1$. The general case reduces to this case by the substitution $x - p = (x_0 - p)y$. Verify!]

16. Prove that if

$$0 < \underline{\lim}\, a_n \leq \overline{\lim}\, a_n < +\infty,$$

then the convergence radius of $\sum a_n(x-p)^n$ is 1.

17. Show that a *conditionally* convergent series $\sum a_n$ $(a_n \in E^1)$ can be rearranged so as to *diverge*, or to converge to *any prescribed sum s*.
[Proof for $s \in E^1$: Using Problem 14(iii), take the first partial sum

$$a_1^+ + \cdots + a_m^+ > s.$$

Then adjoin terms

$$-a_1^-, -a_2^-, \ldots, -a_n^-$$

until the partial sum becomes *less* than s. Then add terms a_k^+ until it exceeds s. Then adjoin terms $-a_k^-$ until it becomes less than s, and so on.

As $a_k^+ \to 0$ and $a_k^- \to 0$ (why?), the rearranged series tends to s. (Why?)

Give a similar proof for $s = \pm\infty$. Also, make the series *oscillate*, with *no* sum.]

18. Prove that if a power series $\sum a_n(x-p)^n$ converges at some $x = x_0 \neq p$, it converges *absolutely* (pointwise) on $G_p(\delta)$ if $\delta \leq |x_0 - p|$.
[Hint: By Theorem 6, $\delta \leq |x_0 - p| \leq r$ (r = convergence radius). Fix any $x \in G_p(\delta)$. Show that the line $\overrightarrow{px}$, when extended, contains a point x_1 such that $|x-p| < |x_1 - p| < \delta \leq r$. By Theorem 6, the series converges *absolutely* at x_1, hence at x as well, by Theorem 7.]

Chapter 5

Differentiation and Antidifferentiation

§1. Derivatives of Functions of One Real Variable

In this chapter, "E" will always denote any one of E^1, E^*, C (the complex field), E^n, *or another normed space. We shall consider functions $f\colon E^1 \to E$ of *one real variable* with values in E. Functions $f\colon E^1 \to E^*$ (admitting finite and infinite values) are said to be *extended real*. Thus $f\colon E^1 \to E$ may be real, extended real, complex, or vector valued.

Operations in E^* were defined in Chapter 4, §4. Recall, in particular, our conventions (2^*) there. Due to them, addition, subtraction, and multiplication are *always* defined in E^* (with sums and products possibly "unorthodox").

To simplify formulations, we shall also adopt the convention that

$$f(x) = 0 \textit{ unless defined otherwise.}$$

("0" stands also for the *zero-vector* in E if E is a vector space.) Thus each function f is defined on *all* of E^1. For convenience, we call $f(x)$ "finite" if $f(x) \neq \pm\infty$ (also if it is a *vector*).

Definition 1.

For each function $f\colon E^1 \to E$, we define its *derived function* $f'\colon E^1 \to E$ by setting, for every point $p \in E^1$,

$$f'(p) = \begin{cases} \lim\limits_{x \to p} \dfrac{f(x) - f(p)}{x - p} & \text{if this limit exists (finite or not);} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Thus $f'(p)$ is *always* defined.

If the *limit* in (1) exists, we call it the *derivative* of f at p.

If, in addition, this limit is finite, we say that f is *differentiable* at p.

If this holds for *each* p in a set $B \subseteq E^1$, we say that f *has a derivative* (respectively, is *differentiable*) *on* B, and we call the function f' the *derivative of* f *on* B.[1]

If the limit in (1) is *one sided* (with $x \to p^-$ or $x \to p^+$), we call it a *one-sided* (*left* or *right*) derivative at p, denoted f'_- or f'_+.

Note that the formula $f'(p) = 0$ holds also if f has *no* derivative at p. On the other hand, $f'(p) \neq 0$ implies that $f'(p)$ is a *genuine* derivative.

Definition 2.

Given a function $f\colon E^1 \to E$, we define its *nth derived function* (or *derived function of order* n), denoted $f^{(n)}\colon E^1 \to E$, by induction:

$$f^{(0)} = f,\ f^{(n+1)} = [f^{(n)}]', \quad n = 0, 1, 2, \ldots .$$

Thus $f^{(n+1)}$ *is the derived function of* $f^{(n)}$. By our conventions, $f^{(n)}$ is defined on all of E^1 for each n and each function $f\colon E^1 \to E$. We have $f^{(1)} = f'$, and we write f'' for $f^{(2)}$, f''' for $f^{(3)}$, etc. We say that f *has* n *derivatives at a point* p iff the limits

$$\lim_{x \to q} \frac{f^{(k)}(x) - f^{(k)}(q)}{x - q}$$

exist for all q in a neighborhood G_p of p and for $k = 0, 1, \ldots, n-2$, and also

$$\lim_{x \to p} \frac{f^{(n-1)}(x) - f^{(n-1)}(p)}{x - p}$$

exists. If all these limits are *finite*, we say that f is n *times differentiable* on I; similarly for one-sided derivatives.

It is an important fact that *differentiability implies continuity*.

Theorem 1. *If a function* $f\colon E^1 \to E$ *is differentiable at a point* $p \in E^1$, *it is continuous at* p, *and* $f(p)$ *is finite* (*even if* $E = E^*$).

Proof. Setting $\Delta x = x - p$ and $\Delta f = f(x) - f(p)$, we have the identity

$$|f(x) - f(p)| = \left| \frac{\Delta f}{\Delta x} \cdot (x - p) \right| \quad \text{for } x \neq p. \tag{2}$$

By assumption,

$$f'(p) = \lim_{x \to p} \frac{\Delta f}{\Delta x}$$

exists and is *finite*. Thus as $x \to p$, the right side of (2) (hence the left side as well) tends to 0, so

$$\lim_{x \to p} |f(x) - f(p)| = 0, \text{ or } \lim_{x \to p} f(x) = f(p),$$

[1] If B is an *interval*, the derivative at its *endpoints* (if *in* B) need be *one sided* only, as $x \to p$ *over* B (see next).

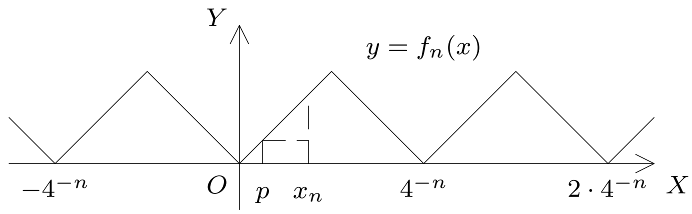

FIGURE 21

proving continuity at p.

Also, $f(p) \neq \pm\infty$, for otherwise $|f(x) - f(p)| = +\infty$ for all x, and so $|f(x) - f(p)|$ cannot tend to 0. □

Note 1. Similarly, the existence of a finite *left* (*right*) derivative at p implies left (right) continuity at p. The proof is the same.

Note 2. The existence of an *infinite* derivative *does not imply continuity, nor does it exclude it*. For example, consider the two cases

(i) $f(x) = \dfrac{1}{x}$, with $f(0) = 0$, and

(ii) $f(x) = \sqrt[3]{x}$.

Give your comments for $p = 0$.

Caution: *A function may be continuous on* E^1 *without being differentiable anywhere* (thus the converse to Theorem 1 *fails*). The first such function was indicated by Weierstrass. We give an example due to Olmsted (Advanced Calculus).

Examples.

(a) We first define a sequence of functions $f_n \colon E^1 \to E^1$ $(n = 1, 2, \ldots)$ as follows. For each $k = 0, \pm 1, \pm 2, \ldots$, let

$$f_n(x) = 0 \text{ if } x = k \cdot 4^{-n}, \text{ and } f_n(x) = \tfrac{1}{2} \cdot 4^{-n} \text{ if } x = (k + \tfrac{1}{2}) \cdot 4^{-n}.$$

Between $k \cdot 4^{-n}$ and $(k \pm \frac{1}{2}) \cdot 4^{-n}$, f_n is *linear* (see Figure 21), so it is continuous on E^1. The series $\sum f_n$ converges *uniformly* on E^1. (Verify!)

Let

$$f = \sum_{n=1}^{\infty} f_n.$$

Then f is continuous on E^1 (why?), yet it is *nowhere differentiable.*

To prove this fact, fix any $p \in E^1$. For each n, let

$$x_n = p + d_n, \text{ where } d_n = \pm 4^{-n-1},$$

choosing the sign of d_n so that p *and* x_n *are in the same half* of a "saw-tooth" in the graph of f_n (Figure 21). Then

$$f_n(x_n) - f_n(p) = \pm d_n = \pm(x_n - p). \quad \text{(Why?)}$$

Also,

$$f_m(x_n) - f_m(p) = \pm d_n \text{ if } m \leq n$$

but *vanishes* for $m > n$. (Why?)

Thus, when computing $f(x_n) - f(p)$, we may replace

$$f = \sum_{m=1}^{\infty} f_m \text{ by } f = \sum_{m=1}^{n} f_m.$$

Since

$$\frac{f_m(x_n) - f_m(p)}{x_n - p} = \pm 1 \text{ for } m \leq n,$$

the fraction

$$\frac{f(x_n) - f(p)}{x_n - p}$$

is an *integer*, odd if n is odd and even if n is even. Thus this fraction *cannot tend to a finite limit as* $n \to \infty$, i.e., as $d_n = 4^{-n-1} \to 0$ and $x_n = p + d_n \to p$. A fortiori, this applies to

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}.$$

Thus f is not differentiable at any p.

The expressions $f(x) - f(p)$ and $x - p$, briefly denoted Δf and Δx, are called the *increments* of f and x (at p), respectively.[2] We now show that for *differentiable* functions, Δf and Δx are "nearly proportional" when x approaches p; that is,

$$\frac{\Delta f}{\Delta x} = c + \delta(x)$$

with c constant and $\lim_{x \to p} \delta(x) = 0$.

Theorem 2. *A function* $f \colon E^1 \to E$ *is differentiable at* p, *and* $f'(p) = c$, *iff there is a finite* $c \in E$ *and a function* $\delta \colon E^1 \to E$ *such that* $\lim_{x \to p} \delta(x) = \delta(p) = 0$, *and such that*

$$\Delta f = [c + \delta(x)]\Delta x \quad \textit{for all } x \in E^1. \tag{3}$$

[2] This notation is rather incomplete but convenient. One only has to remember that both Δf *and* Δx *depend on* x *and* p.

Proof. If f is differentiable at p, put $c = f'(p)$. Define $\delta(p) = 0$ and

$$\delta(x) = \frac{\Delta f}{\Delta x} - f'(p) \text{ for } x \neq p.$$

Then $\lim_{x \to p} \delta(x) = f'(p) - f'(p) = 0 = \delta(p)$. Also, (3) follows.

Conversely, if (3) holds, then

$$\frac{\Delta f}{\Delta x} = c + \delta(x) \to c \text{ as } x \to p \text{ (since } \delta(x) \to 0).$$

Thus by definition,

$$c = \lim_{x \to p} \frac{\Delta f}{\Delta x} = f'(p) \text{ and } f'(p) = c \text{ is } \textit{finite}. \quad \square$$

Theorem 3 (chain rule)**.** *Let the functions* $g\colon E^1 \to E^1$ (*real*) *and* $f\colon E^1 \to E$ (*real or not*) *be differentiable at* p *and* q, *respectively*, *where* $q = g(p)$. *Then the composite function* $h = f \circ g$ *is differentiable at* p, *and*

$$h'(p) = f'(q)g'(p).$$

Proof. Setting

$$\Delta h = h(x) - h(p) = f(g(x)) - f(g(p)) = f(g(x)) - f(q),$$

we must show that

$$\lim_{x \to p} \frac{\Delta h}{\Delta x} = f'(q)g'(p) \neq \pm\infty.$$

Now as f is differentiable at q, Theorem 2 yields a function $\delta\colon E^1 \to E$ such that $\lim_{x \to q} \delta(x) = \delta(q) = 0$ and such that

$$(\forall\, y \in E^1) \quad f(y) - f(q) = [f'(q) + \delta(y)]\Delta y,\ \Delta y = y - q.$$

Taking $y = g(x)$, we get

$$(\forall\, x \in E^1) \quad f(g(x)) - f(q) = [f'(q) + \delta(g(x))][g(x) - g(p)],$$

where

$$g(x) - g(p) = y - q = \Delta y \text{ and } f(g(x)) - f(q) = \Delta h,$$

as noted above. Hence

$$\frac{\Delta h}{\Delta x} = [f'(q) + \delta(g(x))] \cdot \frac{g(x) - g(p)}{x - p} \quad \text{for all } x \neq p.$$

Let $x \to p$. Then we obtain $h'(p) = f'(q)g'(p)$, for, by the continuity of $\delta \circ g$ at p (Chapter 4, §2, Theorem 3),

$$\lim_{x \to p} \delta(g(x)) = \delta(g(p)) = \delta(q) = 0. \quad \square$$

The proofs of the next two theorems are left to the reader.

Theorem 4. *If f, g, and h are real or complex and are differentiable at p, so are*

$$f \pm g,\ hf,\ \textit{and}\ \frac{f}{h}$$

(the latter if $h(p) \neq 0$), and at the point p we have

(i) $(f \pm g)' = f' \pm g'$;

(ii) $(hf)' = hf' + h'f$; *and*

(iii) $\left(\dfrac{f}{h}\right)' = \dfrac{hf' - h'f}{h^2}$.

All this holds also if f and g are vector valued and h is scalar valued. It also applies to infinite (even one-sided) derivatives, except when the limits involved become indeterminate (Chapter 4, §4).

Note 3. By induction, if f, g, and h are n *times* differentiable at a point p, so are $f \pm g$ and hf, and, denoting by $\binom{n}{k}$ the binomial coefficients, we have

(i*) $(f \pm g)^{(n)} = f^{(n)} \pm g^{(n)}$; and

(ii*) $(hf)^{(n)} = \displaystyle\sum_{k=0}^{n} \binom{n}{k} h^{(k)} f^{(n-k)}$.

Formula (ii*) is known as the *Leibniz formula*; its proof is analogous to that of the binomial theorem. It is symbolically written as $(hf)^{(n)} = (h + f)^n$, with the last term interpreted accordingly.[3]

Theorem 5 (componentwise differentiation)**.** *A function $f\colon E^1 \to E^n (C^n)$ is differentiable at p iff each of its n components $(f_1, \ldots, f_n)$ is, and then*

$$f'(p) = (f_1'(p), \ldots, f_n'(p)) = \sum_{k=1}^{n} f_k'(p)\bar{e}_k,$$

with $\bar{e}_k$ as in Theorem 2 of Chapter 3, §§1–3.

In particular, a complex function $f\colon E^1 \to C$ is differentiable iff its real and imaginary parts are, and $f' = f'_{\text{re}} + i \cdot f'_{\text{im}}$ (Chapter 4, §3, Note 5).

Examples (continued)**.**

(b) Consider the *complex exponential*

$$f(x) = \cos x + i \cdot \sin x = e^{xi} \text{ (Chapter 4, §3).}$$

We assume the derivatives of $\cos x$ and $\sin x$ to be known (see Problem 8). By Theorem 5, we have

$$f'(x) = -\sin x + i \cdot \cos x = \cos(x + \tfrac{1}{2}\pi) + i \cdot \sin(x + \tfrac{1}{2}\pi) = e^{(x + \frac{1}{2}\pi)i}.$$

[3] In this connection, recall again the notation introduced in Chapter 4, §3 and also in footnote 1 of Chapter 3, §9 and footnote 1 of Chapter 4, §13. We shall use it throughout.

Hence by induction,

$$f^{(n)}(x) = e^{(x+\frac{1}{2}n\pi)i}, \ n = 1, 2, \ldots. \quad \text{(Verify!)}$$

(c) Define $f\colon E^1 \to E^3$ by

$$f(x) = (1, \cos x, \sin x), \quad x \in E^1.$$

Here Theorem 5 yields

$$f'(p) = (0, -\sin p, \cos p), \quad p \in E^1.$$

For a fixed $p = p_0$, we may consider the line

$$\bar{x} = \bar{a} + t\vec{u},$$

where

$$\bar{a} = f(p_0) \text{ and } \vec{u} = f'(p_0) = (0, -\sin p_0, \cos p_0).$$

This is, by definition, the *tangent vector* at p_0 to the curve $f[E^1]$ in E^3.

More generally, if $f\colon E^1 \to E$ is differentiable at p and continuous on some globe about p, we define the *tangent at p to the curve $f[G_p]$* (in E) to be the line

$$\bar{x} = f(p) + t \cdot f'(p);$$

$f'(p)$ is its direction vector in E, while t is the variable real parameter. For *real* functions $f\colon E^1 \to E^1$, we usually consider not $f[E^1]$ but *the curve $y = f(x)$ in E^2*, i.e., the set

$$\{(x, y) \mid y = f(x), \ x \in E^1\}.$$

The tangent to that curve at p is the line through $(p, f(p))$ with slope $f'(p)$.

In conclusion, let us note that differentiation (i.e., taking derivatives) is a *local limit process* at some point p. Hence (cf. Chapter 4, §1, Note 4) the *existence and the value of $f'(p)$ is not affected by restricting f to some globe G_p about p or by arbitrarily redefining f outside G_p*. For one-sided derivatives, we may replace G_p by its corresponding "half."

Problems on Derived Functions in One Variable

1. Prove Theorems 4 and 5, including (i*) and (ii*). Do it for *dot products* as well.

2. Verify Note 2.

3. Verify Example (a).

3′. Verify Example (b).

4. Prove that if f has finite one-sided derivatives at p, it is continuous at p.

5. Restate and prove Theorems 2 and 3 for *one-sided* derivatives.

6. Prove that if the functions $f_i\colon E^1 \to E^*$ (C) are differentiable at p, so is their product, and

$$(f_1 f_2 \cdots f_m)' = \sum_{i=1}^{m} (f_1 f_2 \cdots f_{i-1} f_i' f_{i+1} \cdots f_m) \text{ at } p.$$

7. A function $f\colon E^1 \to E$ is said to satisfy a *Lipschitz condition* (L) *of order* α $(\alpha > 0)$ *at* p iff

$$(\exists\, \delta > 0)\ (\exists\, K \in E^1)\ (\forall\, x \in G_{\neg p}(\delta)) \quad |f(x) - f(p)| \le K|x - p|^{\alpha}.$$

Prove the following:

(i) This implies continuity at p *but not conversely*; take

$$f(x) = \frac{1}{\ln |x|},\ f(0) = 0.$$

[Hint: For the converse, start with Problem 14(iii) of Chapter 4, §2.]

(ii) L of order $\alpha > 1$ implies differentiability at p, with $f'(p) = 0$.

(iii) Differentiability implies L of order 1, but not conversely. (Take

$$f(x) = x \sin \frac{1}{x},\ f(0) = 0,\ p = 0;$$

then even one-sided derivatives fail to exist.)

8. Let

$$f(x) = \sin x \text{ and } g(x) = \cos x.$$

Show that f and g are differentiable on E^1, with

$$f'(p) = \cos p \text{ and } g'(p) = -\sin p \text{ for each } p \in E^1.$$

Hence prove for $n = 0, 1, 2, \ldots$ that

$$f^{(n)}(p) = \sin\Bigl(p + \frac{n\pi}{2}\Bigr) \text{ and } g^{(n)}(p) = \cos\Bigl(p + \frac{n\pi}{2}\Bigr).$$

[Hint: Evaluate Δf as in Example (d) of Chapter 4, §8. Then use the continuity of f and the formula

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} \frac{z}{\sin z} = 1.$$

To prove the latter, note that

$$|\sin z| \le |z| \le |\tan z|,$$

whence

$$1 \le \frac{z}{\sin z} \le \frac{1}{|\cos z|} \to 1;$$

similarly for g.]

9. Prove that if f is differentiable at p then

$$f'(p) = \lim_{\substack{x \to p^+ \\ y \to p^-}} \frac{f(x) - f(y)}{x - y} \neq \pm\infty;$$

i.e., $(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x \in (p, p + \delta))\ (\forall y \in (p - \delta, p))$

$$\left| \frac{f(x) - f(y)}{x - y} - f'(p) \right| < \varepsilon.$$

Disprove the converse by *redefining f at p* (note that the above limit *does not* involve $f(p)$).
[Hint: If $y < p < x$ then

$$\begin{aligned} \left| \frac{f(x) - f(y)}{x - y} - f'(p) \right| &\leq \left| \frac{f(x) - f(p)}{x - y} - \frac{x - p}{x - y} f'(p) \right| + \left| \frac{f(p) - f(y)}{x - y} - \frac{p - y}{x - y} f'(p) \right| \\ &\leq \left| \frac{f(x) - f(p)}{x - p} - f'(p) \right| + \left| \frac{f(p) - f(y)}{p - y} - f'(p) \right| \to 0.] \end{aligned}$$

10. Prove that if f is twice differentiable at p, then

$$f''(x) = \lim_{h \to 0} \frac{f(p + h) - 2f(p) + f(p - h)}{h^2} \neq \pm\infty.$$

Does the converse hold (cf. Problem 9)?

11. In Example (c), find the three *coordinate* equations of the tangent line at $p = \frac{1}{2}\pi$.

12. Judging from Figure 22 in §2, discuss the existence, finiteness, and sign of the derivatives (or one-sided derivatives) of f at the points p_i indicated.

13. Let $f\colon E^n \to E$ be *linear*, i.e., such that

$$(\forall \bar{x}, \bar{y} \in E^n)\ (\forall a, b \in E^1) \quad f(a\bar{x} + b\bar{y}) = af(\bar{x}) + bf(\bar{y}).$$

Prove that if $g\colon E^1 \to E^n$ is differentiable at p, so is $h = f \circ g$ and $h'(p) = f(g'(p))$.
[Hint: f is *continuous* since $f(\bar{x}) = \sum_{k=1}^{n} x_k f(\bar{e}_k)$. See Problem 5 in Chapter 3, §§4–6.]

§2. Derivatives of Extended-Real Functions

For a while (in §§2 and 3), we limit ourselves to extended-real functions. Below, f and g are *real or extended real* ($f, g\colon E^1 \to E^*$). We assume, however, that they are *not constantly infinite* on any interval (a, b), $a < b$.

Lemma 1. *If $f'(p) > 0$ at some $p \in E^1$, then*

$$x < p < y$$

implies

$$f(x) < f(p) < f(y)$$

for all x, y in a sufficiently small globe $G_p(\delta) = (p - \delta, p + \delta)$.[1]

Similarly, if $f'(p) < 0$, then $x < p < y$ implies $f(x) > f(p) > f(y)$ for x, y in some $G_p(\delta)$.

Proof. If $f'(p) > 0$, the "0" case in Definition 1 of §1, is excluded, so

$$f'(p) = \lim_{x \to p} \frac{\Delta f}{\Delta x} > 0.$$

Hence we must also have $\Delta f / \Delta x > 0$ for x in some $G_p(\delta)$.

It follows that Δf and Δx have *the same sign* in $G_p(\delta)$; i.e.,

$$f(x) - f(p) > 0 \text{ if } x > p \text{ and } f(x) - f(p) < 0 \text{ if } x < p.$$

(This implies $f(p) \neq \pm\infty$. Why?) Hence

$$x < p < y \Longrightarrow f(x) < f(p) < f(y),$$

as claimed; similarly in case $f'(p) < 0$. □

Corollary 1. *If $f(p)$ is the maximum or minimum value of $f(x)$ for x in some $G_p(\delta)$, then $f'(p) = 0$; i.e., f has a zero derivative, or none at all, at p.*

For, by Lemma 1, $f'(p) \neq 0$ *excludes* a maximum or minimum at p. (Why?)

Note 1. Thus $f'(p) = 0$ is a *necessary* condition for a local maximum or minimum at p. It is *insufficient*, however. For example, if $f(x) = x^3$, f has no maxima or minima at all, yet $f'(0) = 0$. For sufficient conditions, see §6.

Figure 22 illustrates these facts at the points $p_2, p_3, \ldots, p_{11}$. Note that in Figure 22, the isolated points P, Q, R *belong* to the graph.

Geometrically, $f'(p) = 0$ means that the tangent at p is horizontal, or that a two-sided tangent does not exist at p.

Theorem 1. *Let $f\colon E^1 \to E^*$ be relatively continuous on an interval $[a, b]$, with $f' \neq 0$ on (a, b). Then f is strictly monotone on $[a, b]$, and f' is sign-constant there (possibly 0 at a and b), with $f' \geq 0$ if $f\uparrow$, and $f' \leq 0$ if $f\downarrow$.*

Proof. By Theorem 2 of Chapter 4, §8, f attains a *least* value m, and a *largest* value M, at some points of $[a, b]$. However, neither can occur at an *interior* point $p \in (a, b)$, for, by Corollary 1, this would imply $f'(p) = 0$, contrary to our assumption.

[1] This does not mean that f is *monotone* on any G_p (see Problem 6). We shall only say in such cases that f increases *at the point* p.

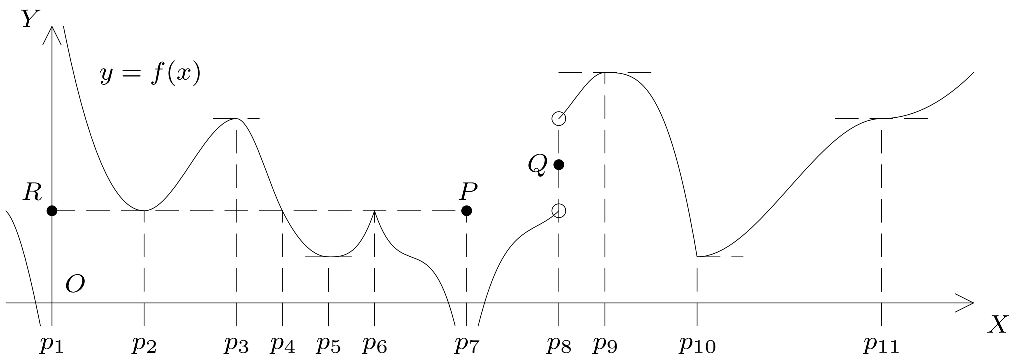

Figure 22

Thus $M = f(a)$ or $M = f(b)$; for the moment we assume $M = f(b)$ and $m = f(a)$. We must have $m < M$, for $m = M$ would make f *constant* on $[a, b]$, implying $f' = 0$. Thus $m = f(a) < f(b) = M$.

Now let $a \leq x < y \leq b$. Applying the previous argument to each of the intervals $[a, x]$, $[a, y]$, $[x, y]$, and $[x, b]$ (now using that $m = f(a) < f(b) = M$), we find that

$$f(a) \leq f(x) < f(y) \leq f(b). \quad \text{(Why?)}$$

Thus $a \leq x < y \leq b$ *implies* $f(x) < f(y)$; i.e., f increases on $[a, b]$. Hence f' cannot be *negative* at any $p \in [a, b]$, for, otherwise, by Lemma 1, f would *decrease* at p. Thus $f' \geq 0$ on $[a, b]$.

In the case $M = f(a) > f(b) = m$, we would obtain $f' \leq 0$. $\square$

Caution: The function f may increase or decrease at p even if $f'(p) = 0$. See Note 1.

Corollary 2 (Rolle's theorem)**.** *If $f: E^1 \to E^*$ is relatively continuous on $[a, b]$ and if $f(a) = f(b)$, then $f'(p) = 0$ for at least one interior point $p \in (a, b)$.*

For, if $f' \neq 0$ on *all* of (a, b), then by Theorem 1, f would be *strictly* monotone on $[a, b]$, so the equality $f(a) = f(b)$ would be impossible.

Figure 22 illustrates this on the intervals $[p_2, p_4]$ and $[p_4, p_6]$, with $f'(p_3) = f'(p_5) = 0$. A discontinuity at 0 causes an apparent failure on $[0, p_2]$.

Note 2. Theorem 1 and Corollary 2 hold even if $f(a)$ and $f(b)$ are infinite, if continuity is interpreted in the sense of the metric ρ' of Problem 5 in Chapter 3, §11. (Weierstrass' Theorem 2 of Chapter 4, §8 applies to (E^*, ρ'), with the same proof.)

Theorem 2 (Cauchy's law of the mean)**.** *Let the functions $f, g: E^1 \to E^*$ be relatively continuous and finite on $[a, b]$ and have derivatives on (a, b), with f' and g' never* both *infinite* at the same point $p \in (a, b)$. *Then*

$$g'(q)[f(b) - f(a)] = f'(q)[g(b) - g(a)] \text{ for at least one } q \in (a, b). \tag{1}$$

Proof. Let $A = f(b) - f(a)$ and $B = g(b) - g(a)$. We must show that $Ag'(q) = Bf'(q)$ for some $q \in (a, b)$. For this purpose, consider the function $h = Ag - Bf$. It is relatively continuous and finite on $[a, b]$, as are g and f. Also,

$$h(a) = f(b)g(a) - g(b)f(a) = h(b). \quad \text{(Verify!)}$$

Thus by Corollary 2, $h'(q) = 0$ for some $q \in (a, b)$. Here, by Theorem 4 of §1, $h' = (Ag - Bf)' = Ag' - Bf'$. (This is legitimate, for, by assumption, f' and g' never *both* become infinite, so no indeterminate limits occur.) Thus $h'(q) = Ag'(q) - Bf'(q) = 0$, and (1) follows. □

Corollary 3 (Lagrange's law of the mean)**.** *If $f \colon E^1 \to E^1$ is relatively continuous on $[a, b]$ with a derivative on (a, b), then*

$$f(b) - f(a) = f'(q)(b - a) \textit{ for at least one } q \in (a, b). \tag{2}$$

Proof. Take $g(x) = x$ in Theorem 2, so $g' = 1$ on E^1. □

Note 3. Geometrically,

$$\frac{f(b) - f(a)}{b - a}$$

is the slope of the *secant* through $(a, f(a))$ and $(b, f(b))$, and $f'(q)$ is the slope of the tangent line at q. Thus Corollary 3 states that *the secant is parallel to the tangent at some intermediate point q*; see Figure 23. Theorem 2 states the same for curves given *parametrically*: $x = f(t)$, $y = g(t)$.

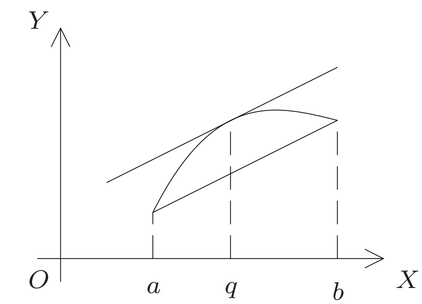

FIGURE 23

Corollary 4. *Let f be as in Corollary 3. Then*

(i) *f is constant on $[a, b]$ iff $f' = 0$ on (a, b);*

(ii) *$f\uparrow$ on $[a, b]$ iff $f' \geq 0$ on (a, b); and*

(iii) *$f\downarrow$ on $[a, b]$ iff $f' \leq 0$ on (a, b).*

Proof. Let $f' = 0$ on (a, b). If $a \leq x \leq y \leq b$, apply Corollary 3 to the interval $[x, y]$ to obtain

$$f(y) - f(x) = f'(q)(y - x) \text{ for some } q \in (a, b) \text{ and } f'(q) = 0.$$

Thus $f(y) - f(x) = 0$ for $x, y \in [a, b]$, so f is constant.

The rest is left to the reader. □

Theorem 3 (inverse functions)**.** *Let $f\colon E^1 \to E^1$ be relatively continuous and strictly monotone on an interval $I \subseteq E^1$. Let $f'(p) \neq 0$ at some* interior *point $p \in I$. Then the inverse function $g = f^{-1}$ (with f restricted to I) has a derivative at $q = f(p)$, and*

$$g'(q) = \frac{1}{f'(p)}.$$

(If $f'(p) = \pm\infty$, then $g'(q) = 0$.)

Proof. By Theorem 3 of Chapter 4, §9, $g = f^{-1}$ is strictly monotone and relatively continuous on $f[I]$, itself an interval. If p is interior to I, then $q = f(p)$ is interior to $f[I]$. (Why?)

Now if $y \in f[I]$, we set

$$\Delta g = g(y) - g(q),\ \Delta y = y - q,\ x = f^{-1}(y) = g(y), \text{ and } f(x) = y$$

and obtain

$$\frac{\Delta g}{\Delta y} = \frac{g(y) - g(q)}{y - q} = \frac{x - p}{f(x) - f(p)} = \frac{\Delta x}{\Delta f} \text{ for } x \neq p.$$

Now if $y \to q$, the continuity of g at q yields $g(y) \to g(q)$; i.e., $x \to p$. Also, $x \neq p$ iff $y \neq q$, for f and g are one-to-one functions. Thus we may substitute $y = f(x)$ or $x = g(y)$ to get

$$g'(q) = \lim_{y \to q} \frac{\Delta g}{\Delta y} = \lim_{x \to p} \frac{\Delta x}{\Delta f} = \frac{1}{\lim\limits_{x \to p} (\Delta f / \Delta x)} = \frac{1}{f'(p)},^{2} \tag{2'}$$

where we use the convention $\frac{1}{\infty} = 0$ if $f'(p) = \infty$. □

Examples.

(A) Let

$$f(x) = \log_a |x| \text{ with } f(0) = 0.$$

Let $p > 0$. Then $(\forall\, x > 0)$

$$\begin{aligned} \Delta f = f(x) - f(p) &= \log_a x - \log_a p = \log_a (x/p) \\ &= \log_a \frac{p + (x - p)}{p} = \log_a \Big(1 + \frac{\Delta x}{p}\Big). \end{aligned}$$

Thus

$$\frac{\Delta f}{\Delta x} = \log_a \Big(1 + \frac{\Delta x}{p}\Big)^{1/\Delta x}.$$

[2] More precisely, we are replacing the x by $g(y)$ in $(x - p)/[f(x) - f(p)]$ by Corollary 2 of Chapter 4, §2 to obtain $g'(q)$. The steps in $(2')$ should be *reversed*.

Now let $z = \Delta x/p$. (Why is this substitution admissible?) Then using the formula

$$\lim_{z\to 0}(1+z)^{1/z} = e \quad \text{(see Chapter 4, §2, Example (C))}$$

and the continuity of the log and power functions, we obtain

$$f'(p) = \lim_{x\to p}\frac{\Delta f}{\Delta x} = \lim_{z\to 0}\log_a[(1+z)^{1/z}]^{1/p} = \log_a e^{1/p} = \frac{1}{p}\log_a e.$$

The same formula results also if $p < 0$, i.e., $|p| = -p$. At $p = 0$, f has one-sided derivatives ($\pm\infty$) only (verify!), so $f'(0) = 0$ by Definition 1 in §1.

(B) *The inverse of the* $\log_a$ function is the *exponential* $g\colon E^1 \to E^1$, with

$$g(y) = a^y \quad (a > 0,\ a \neq 1).$$

By Theorem 3, we have

$$(\forall\, q \in E^1) \quad g'(q) = \frac{1}{f'(p)},\ p = g(q) = a^q.$$

Thus

$$g'(q) = \frac{1}{\frac{1}{p}\log_a e} = \frac{p}{\log_a e} = \frac{a^q}{\log_a e}.$$

Symbolically,

$$(\log_a|x|)' = \frac{1}{x}\log_a e\ (x \neq 0); \quad (a^x)' = \frac{a^x}{\log_a e} = a^x \ln a. \tag{3}$$

In particular, if $a = e$, we have $\log_e a = 1$ and $\log_a x = \ln x$; hence

$$(\ln|x|)' = \frac{1}{x}\ (x \neq 0) \quad \text{and} \quad (e^x)' = e^x \quad (x \in E^1). \tag{4}$$

(C) The *power function* $g\colon (0, +\infty) \to E^1$ is given by

$$g(x) = x^a = \exp(a \cdot \ln x) \text{ for } x > 0 \text{ and fixed } a \in E^1.$$

By the chain rule (§1, Theorem 3), we obtain

$$g'(x) = \exp(a \cdot \ln x)\cdot\frac{a}{x} = x^a \cdot \frac{a}{x} = a \cdot x^{a-1}.$$

Thus we have the symbolic formula

$$(x^a)' = a \cdot x^{a-1} \text{ for } x > 0 \text{ and fixed } a \in E^1. \tag{5}$$

Theorem 4 (Darboux). *If $f\colon E^1 \to E^*$ is relatively continuous and has a derivative on an interval I, then f' has the Darboux property* (*Chapter* 4, §9) *on I.*

Proof. Let $p, q \in I$ and $f'(p) < c < f'(q)$. Put $g(x) = f(x) - cx$. Assume $g' \neq 0$ on (p, q) and find a contradiction to Theorem 1. Details are left to the reader. □

Problems on Derivatives of Extended-Real Functions

1. Complete the missing details in the proof of Theorems 1, 2, and 4, Corollary 4, and Lemma 1.
[Hint for converse to Corollary 4(ii): Use Lemma 1 for an indirect proof.]

2. Do cases $p \leq 0$ in Example (A).

3. Show that Theorems 1, 2, and 4 and Corollaries 2 to 4 hold also if f is discontinuous at a and b but $f(a^+)$ and $f(b^-)$ exist and are finite. (In Corollary 2, assume also $f(a^+) = f(b^-)$; in Theorems 1 and 4 and Corollary 2, finiteness is unnecessary.)
[Hint: Redefine $f(a)$ and $f(b)$.]

4. Under the assumptions of Corollary 3, show that f' cannot *stay* infinite on any interval (p, q), $a \leq p < q \leq b$.
[Hint: Apply Corollary 3 to the interval $[p, q]$.]

5. Justify footnote 1.
[Hint: Let
$$f(x) = x + 2x^2 \sin \frac{1}{x^2} \text{ with } f(0) = 0.$$
At 0, find f' from Definition 1 in §1. Use also Problem 8 of §1. Show that f is not monotone on any $G_0(\delta)$.]

6. Show that f' need not be continuous or bounded on $[a, b]$ (under the standard metric), even if f is differentiable there.
[Hint: Take f as in Problem 5.]

7. With f as in Corollaries 3 and 4, prove that if $f' \geq 0$ ($f' \leq 0$) on (a, b) and if f' is not *constantly* 0 on any subinterval $(p, q) \neq \emptyset$, then f is *strictly* monotone on $[a, b]$.

8. Let $x = f(t)$, $y = g(t)$, where t varies over an open interval $I \subseteq E^1$, define a curve in E^2 parametrically. Prove that if f and g have derivatives on I and $f' \neq 0$, then the function $h = f^{-1}$ has a derivative on $f[I]$, and the slope of the tangent to the curve at t_0 equals $g'(t_0)/f'(t_0)$.
[Hint: The word "curve" implies that f and g are continuous on I (Chapter 4, §10), so Theorems 1 and 3 apply, and $h = f^{-1}$ is a *function*. Also, $y = g(h(x))$. Use Theorem 3 of §1.]

9. Prove that if f is continuous and has a derivative on (a, b) and if f' has a finite or infinite (even one-sided) limit at some $p \in (a, b)$, then this limit equals $f'(p)$. Deduce that f' is continuous at p if $f'(p^-)$ and $f'(p^+)$ exist.

[Hint: By Corollary 3, for each $x \in (a, b)$, there is some q_x *between* p and x such that

$$f'(q_x) = \frac{\Delta f}{\Delta x} \to f'(p) \text{ as } x \to p.$$

Set $y = q_x$, so $\lim_{y \to p} f'(y) = f'(p)$.]

10. From Theorem 3 and Problem 8 in §1, deduce the differentiation formulas

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}\,;\ (\arccos x)' = \frac{-1}{\sqrt{1-x^2}}\,;\ (\arctan x)' = \frac{1}{1+x^2}.$$

11. Prove that if f has a derivative at p, then $f(p)$ is *finite*, provided f is not constantly infinite on any interval (p, q) or (q, p), $p \neq q$.

[Hint: If $f(p) = \pm\infty$, each G_p has points at which $\frac{\Delta f}{\Delta x} = +\infty$, *as well as* those x with $\frac{\Delta f}{\Delta x} = -\infty$.]

§3. L'Hôpital's Rule

We shall now prove a useful rule for resolving indeterminate limits. Below, $G_{\neg p}$ denotes a deleted globe $G_{\neg p}(\delta)$ in E^1, or one about $\pm\infty$ of the form $(a, +\infty)$ or $(-\infty, a)$. For one-sided limits, replace $G_{\neg p}$ by its appropriate "half."

Theorem 1 (L'Hôpital's rule)**.** *Let $f, g \colon E^1 \to E^*$ be differentiable on $G_{\neg p}$, with $g' \neq 0$ there. If $|f(x)|$ and $|g(x)|$ tend both to $+\infty$,*[1] *or both to 0, as $x \to p$ and if*

$$\lim_{x \to p} \frac{f'(x)}{g'(x)} = r \text{ exists in } E^*,$$

then also

$$\lim_{x \to p} \frac{f(x)}{g(x)} = r;$$

similarly for $x \to p^+$ or $x \to p^-$.

Proof. It suffices to consider *left* and *right* limits. Both combined then yield the two-sided limit.

First, let $-\infty \leq p < +\infty$,

$$\lim_{x \to p^+} |f(x)| = \lim_{x \to p^+} |g(x)| = +\infty \text{ and } \lim_{x \to p^+} \frac{f'(x)}{g'(x)} = r \ (\textit{finite}).$$

[1] This includes the cases $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$.

Then given $\varepsilon > 0$, we can fix $a > p$ ($a \in G_{\neg p}$) such that

$$\left|\frac{f'(x)}{g'(x)} - r\right| < \varepsilon, \text{ for all } x \text{ in the interval } (p, a). \tag{1}$$

Now apply Cauchy's law of the mean (§2, Theorem 2) to each interval $[x, a]$, $p < x < a$. This yields, for each such x, some $q \in (x, a)$ with

$$g'(q)[f(x) - f(a)] = f'(q)[g(x) - g(a)].$$

As $g' \neq 0$ (by assumption), $g(x) \neq g(a)$ by Theorem 1, §2, so we may divide to obtain

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(q)}{g'(q)}, \quad \text{where } p < x < q < a.$$

This combined with (1) yields

$$\left|\frac{f(x) - f(a)}{g(x) - g(a)} - r\right| < \varepsilon,$$

or, setting

$$F(x) = \frac{1 - f(a)/f(x)}{1 - g(a)/g(x)},$$

we have

$$\left|\frac{f(x)}{g(x)} \cdot F(x) - r\right| < \varepsilon \text{ for all } x \text{ inside } (p, a). \tag{2}$$

As $|f(x)|$ and $|g(x)| \to +\infty$ (by assumption), we have $F(x) \to 1$ as $x \to p^+$. Hence by rules for right limits, there is $b \in (p, a)$ such that for all $x \in (p, b)$, both $|F(x) - 1| < \varepsilon$ and $F(x) > \frac{1}{2}$. (Why?) For such x, formula (2) holds as well. Also,

$$\frac{1}{|F(x)|} < 2 \text{ and } |r - rF(x)| = |r|\,|1 - F(x)| < |r|\,\varepsilon.$$

Combining this with (2), we have for $x \in (p, b)$

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - r\right| &= \frac{1}{|F(x)|}\left|\frac{f(x)}{g(x)}F(x) - rF(x)\right| \\
&< 2\left|\frac{f(x)}{g(x)} \cdot F(x) - r + r - rF(x)\right| \\
&< 2\varepsilon(1 + |r|).
\end{aligned}$$

Thus, given $\varepsilon > 0$, we found $b > p$ such that

$$\left|\frac{f(x)}{g(x)} - r\right| < 2\varepsilon(1 + |r|), \quad x \in (p, b).$$

As ε is *arbitrary*, we have $\lim_{x\to p^+} \frac{f(x)}{g(x)} = r$, as claimed.

The case $\lim_{x\to p^+} f(x) = \lim_{x\to p^+} g(x) = 0$ is simpler. As before, we obtain

$$\left|\frac{f(x)-f(a)}{g(x)-g(a)} - r\right| < \varepsilon.$$

Here we may as well replace "a" by any $y \in (p,\, a)$. Keeping y fixed, let $x \to p^+$. Then $f(x) \to 0$ and $g(x) \to 0$, so we get

$$\left|\frac{f(y)}{g(y)} - r\right| \le \varepsilon \text{ for any } y \in (p,\, a).$$

As ε is arbitrary, this implies $\lim_{y\to p^+} \frac{f(y)}{g(y)} = r$. Thus the case $x \to p^+$ is settled for a *finite* r.

The cases $r = \pm\infty$ and $x \to p^-$ are analogous, and we leave them to the reader. □

Note 1. $\lim \frac{f(x)}{g(x)}$ may exist even if $\lim \frac{f'(x)}{g'(x)}$ does not. For example, take

$$f(x) = x + \sin x \text{ and } g(x) = x.$$

Then

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty}\left(1 + \frac{\sin x}{x}\right) = 1 \quad \text{(why?)},$$

but

$$\frac{f'(x)}{g'(x)} = 1 + \cos x$$

does not tend to any limit as $x \to +\infty$.

Note 2. The rule fails if the required assumptions are not satisfied, e.g., if g' has zero values in *each* $G_{\neg p}$; see Problem 4 below.

Often it is useful to combine L'Hôpital's rule with some known limit formulas, such as

$$\lim_{z\to 0}(1+z)^{1/z} = e \text{ or } \lim_{x\to 0} \frac{x}{\sin x} = 1 \text{ (see §1, Problem 8).}$$

Examples.

(a) $\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{(\ln x)'}{1} = \lim_{x\to+\infty} \frac{1}{x} = 0.$

(b) $\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1/(1+x)}{1} = 1.$

(c) $\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{6} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{6}.$

(Here we had to apply L'Hôpital's rule *repeatedly*.)

(d) Consider

$$\lim_{x\to 0^+} \frac{e^{-1/x}}{x}.$$

Taking derivatives (even n times), one gets

$$\lim_{x\to 0^+} \frac{e^{-1/x}}{n!\,x^{n+1}}, \quad n = 1, 2, 3, \ldots \text{ (always indeterminate!).}$$

Thus the rule gives no results. In this case, however, a simple device helps (see Problem 5 below).

(e) $\lim_{n\to\infty} n^{1/n}$ does not have the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so the rule does not apply directly. Instead we compute

$$\lim_{n\to\infty} \ln n^{1/n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0 \text{ (Example (a)).}$$

Hence

$$n^{1/n} = \exp(\ln n^{1/n}) \to \exp(0) = e^0 = 1$$

by the continuity of exponential functions. The answer is then 1.

Problems on L'Hôpital's Rule

Elementary differentiation formulas are assumed known.

1. Complete the proof of L'Hôpital's rule. Verify that the differentiability assumption may be replaced by continuity plus existence of finite or infinite (but not *both together* infinite) derivatives f' and g' on $G_{\neg p}$ (same proof).

2. Show that the rule fails for *complex* functions. See, however, Problems 3, 7, and 8.

[Hint: Take $p = 0$ with

$$f(x) = x \text{ and } g(x) = x + x^2 e^{i/x^2} = x + x^2\left(\cos\frac{1}{x^2} + i\cdot\sin\frac{1}{x^2}\right).$$

Then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = 1, \text{ though } \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{1}{g'(x)} = 0.$$

Indeed, $g'(x) - 1 = (2x - 2i/x)e^{i/x^2}$. (Verify!) Hence

$$|g'(x)| + 1 \geq |2x - 2i/x| \quad (\text{for } |e^{i/x^2}| = 1),$$

so

$$|g'(x)| \geq -1 + \frac{2}{x}. \quad \text{(Why?)}$$

Deduce that

$$\left|\frac{1}{g'(x)}\right| \leq \left|\frac{x}{2-x}\right| \to 0.]$$

3. Prove the "*simplified rule of L'Hôpital*" for real or complex functions (also for vector-valued f and scalar-valued g): If f and g are differentiable at p, with $g'(p) \neq 0$ and $f(p) = g(p) = 0$, then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{f'(p)}{g'(p)}.$$

[Hint:

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(p)}{g(x) - g(p)} = \frac{\Delta f}{\Delta x} \Big/ \frac{\Delta g}{\Delta x} \to \frac{f'(p)}{g'(p)}.]$$

4. Why does $\lim_{x \to +\infty} \dfrac{f(x)}{g(x)}$ not exist, though $\lim_{x \to +\infty} \dfrac{f'(x)}{g'(x)}$ *does*, in the following example? Verify and explain.

$$f(x) = e^{-2x}(\cos x + 2 \sin x), \quad g(x) = e^{-x}(\cos x + \sin x).$$

[Hint: g' vanishes many times in each $G_{+\infty}$. Use the Darboux property for the proof.]

5. Find $\lim_{x \to 0^+} \dfrac{e^{-1/x}}{x}$.

[Hint: Substitute $z = \frac{1}{x} \to +\infty$. *Then* use the rule.]

6. Verify that the assumptions of L'Hôpital's rule hold, and find the following limits.

(a) $\lim_{x \to 0} \dfrac{e^x - e^{-x}}{\ln(e - x) + x - 1}$;

(b) $\lim_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$;

(c) $\lim_{x \to 0} \dfrac{(1 + x)^{1/x} - e}{x}$;

(d) $\lim_{x \to 0^+} (x^q \ln x)$, $q > 0$;

(e) $\lim_{x \to +\infty} (x^{-q} \ln x)$, $q > 0$;

(f) $\lim_{x \to 0^+} x^x$;

(g) $\lim_{x \to +\infty} (x^q a^{-x})$, $a > 1$, $q > 0$;

(h) $\lim_{x \to 0} \left(\dfrac{1}{x^2} - \operatorname{cotan}^2 x \right)$;

(i) $\lim_{x \to +\infty} \left(\dfrac{\pi}{2} - \arctan x \right)^{1/\ln x}$;

(j) $\lim_{x \to 0} \left(\dfrac{\sin x}{x} \right)^{1/(1 - \cos x)}$.

7. Prove L'Hôpital's rule for $f\colon E^1 \to E^n$ (C) and $g\colon E^1 \to E^1$, with

$$\lim_{k\to p} |f(x)| = 0 = \lim_{x\to p} |g(x)|,\ p \in E^* \text{ and } r \in E^n,$$

leaving the other assumptions unchanged.

[Hint: Apply the rule to the *components* of $\frac{f}{g}$ (respectively, to $\left(\frac{f}{g}\right)_{\text{re}}$ and $\left(\frac{f}{g}\right)_{\text{im}}$).]

8. Let f and g be complex and differentiable on $G_{\neg p}$, $p \in E^1$. Let

$$\lim_{x\to p} f(x) = \lim_{x\to p} g(x) = 0,\ \lim_{x\to p} f'(x) = q, \text{ and } \lim_{x\to p} g'(x) = r \neq 0.$$

Prove that $\lim_{x\to p} \dfrac{f(x)}{g(x)} = \dfrac{q}{r}$.

[Hint:

$$\frac{f(x)}{g(x)} = \frac{f(x)}{x-p} \Big/ \frac{g(x)}{x-p}.$$

Apply Problem 7 to find

$$\lim_{x\to p} \frac{f(x)}{x-p} \text{ and } \lim_{x\to p} \frac{g(x)}{x-p}.]$$

***9.** Do Problem 8 for $f\colon E^1 \to C^n$ and $g\colon E^1 \to C$.

§4. Complex and Vector-Valued Functions on E^1

The theorems of §§2–3 *fail* for complex and vector-valued functions (see Problem 3 below and Problem 2 in §3). However, some analogues hold. In a sense, they even are stronger, for, unlike the previous theorems, they do not require the existence of a derivative *on an entire interval* $I \subseteq E^1$, but only on $I - Q$, where Q is a *countable set*, i.e., one contained in the range of a sequence, $Q \subseteq \{p_m\}$. (We henceforth presuppose §9 of Chapter 1.)

In the following theorem, due to N. Bourbaki,[1] $g\colon E^1 \to E^*$ is *extended real* while f may also be complex or vector valued. We call it the *finite increments law* since it deals with "finite increments" $f(b)-f(a)$ and $g(b)-g(a)$. Roughly, it states that $|f'| \le g'$ implies a similar inequality for increments.

Theorem 1 (finite increments law). *Let $f\colon E^1 \to E$ and $g\colon E^1 \to E^*$ be relatively continuous and finite on a closed interval $I = [a, b] \subseteq E^1$, and have derivatives*[2] *with $|f'| \le g'$, on $I - Q$ where $Q \subseteq \{p_1, p_2, \ldots, p_m, \ldots\}$. Then*

$$|f(b) - f(a)| \le g(b) - g(a). \tag{1}$$

[1] This is the pen name of a famous school of twentieth-century French mathematicians.

[2] Actually, *right* derivatives suffice, as will follow from the proof. (*Left* derivatives suffice as well.)

The proof is somewhat laborious, but worthwhile. (At a first reading, one may omit it, however.) We outline some preliminary ideas.

Given any $x \in I$, suppose first that $x > p_m$ for at least one $p_m \in Q$. In this case, we put

$$Q(x) = \sum_{p_m < x} 2^{-m};$$

here the summation is only over those m for which $p_m < x$. If, however, there are no $p_m \in Q$ with $p_m < x$, we put $Q(x) = 0$. Thus $Q(x)$ is defined for all $x \in I$. It gives an idea as to "how many" p_m (at which f may have no derivative) *precede* x. Note that $x < y$ implies $Q(x) \leq Q(y)$. (Why?) Also,

$$Q(x) \leq \sum_{m=1}^{\infty} 2^{-m} = 1.$$

Our plan is as follows. To prove (1), it suffices to show that for some *fixed* $K \in E^1$, we have

$$(\forall\, \varepsilon > 0) \quad |f(b) - f(a)| \leq g(b) - g(a) + K\varepsilon,$$

for then, letting $\varepsilon \to 0$, we obtain (1). We choose

$$K = b - a + Q(b), \text{ with } Q(x) \text{ as above.}$$

Temporarily fixing $\varepsilon > 0$, let us call a point $r \in I$ "good" iff

$$|f(r) - f(a)| \leq g(r) - g(a) + [r - a + Q(r)]\varepsilon \tag{2}$$

and "bad" otherwise. We shall show that b is "good." First, we prove a lemma.

Lemma 1. *Every "good" point $r \in I$ $(r < b)$ is followed by a whole interval (r, s), $r < s \leq b$, consisting of "good" points only.*

Proof. First let $r \notin Q$, so by assumption, f and g have derivatives at r, with

$$|f'(r)| \leq g'(r).$$

Suppose $g'(r) < +\infty$. Then (treating g' as a *right* derivative) we can find $s > r$ $(s \leq b)$ such that, for all x in the interval (r, s),

$$\left|\frac{g(x) - g(r)}{x - r} - g'(r)\right| < \frac{\varepsilon}{2} \quad \text{(why?);}$$

similarly for f. Multiplying by $x - r$, we get

$$|f(x) - f(r) - f'(r)(x - r)| < (x - r)\frac{\varepsilon}{2} \text{ and}$$
$$|g(x) - g(r) - g'(r)(x - r)| < (x - r)\frac{\varepsilon}{2},$$

and hence by the triangle inequality (explain!),

$$|f(x) - f(r)| \le |f'(r)|(x - r) + (x - r)\frac{\varepsilon}{2}$$

and

$$g'(r)(x - r) + (x - r)\frac{\varepsilon}{2} < g(x) - g(r) + (x - r)\varepsilon.$$

Combining this with $|f'(r)| \le g'(r)$, we obtain

$$|f(x) - f(r)| \le g(x) - g(r) + (x - r)\varepsilon \text{ whenever } r < x < s. \tag{3}$$

Now as r is "good," it satisfies (2); hence, certainly, as $Q(r) \le Q(x)$,

$$|f(r) - f(a)| \le g(r) - g(a) + (r - a)\varepsilon + Q(x)\varepsilon \text{ whenever } r < x < s.$$

Adding this to (3) and using the triangle inequality again, we have

$$|f(x) - f(a)| \le g(x) - g(a) + [x - a + Q(x)]\varepsilon \text{ for all } x \in (r, s).$$

By definition, this shows that each $x \in (r, s)$ is "good," as claimed. Thus the lemma is proved for the case $r \in I - Q$, with $g'(r) < +\infty$.

The cases $g'(r) = +\infty$ and $r \in Q$ are left as Problems 1 and 2. □

We now return to Theorem 1.

Proof of Theorem 1. Seeking a contradiction, suppose b is "bad," and let $B \neq \emptyset$ be the set of all "bad" points in $[a, b]$. Let

$$r = \inf B, \quad r \in [a, b].$$

Then the interval $[a, r)$ can contain only "good" points, i.e., points x such that

$$|f(x) - f(a)| \le g(x) - g(a) + [x - a + Q(x)]\varepsilon.$$

As $x < r$ implies $Q(x) \le Q(r)$, we have

$$|f(x) - f(a)| \le g(x) - g(a) + [x - a + Q(r)]\varepsilon \text{ for all } x \in [a, r). \tag{4}$$

Note that $[a, r) \neq \emptyset$, for by (2), a is certainly "good" (why?), and so Lemma 1 yields a whole interval $[a, s)$ of "good" points contained in $[a, r)$.

Letting $x \to r$ in (4) and using the continuity of f at r, we obtain (2). Thus r is "good" itself. Then, however, Lemma 1 yields a *new* interval (r, q) of "good" points. Hence $[a, q)$ has no "bad" points, and so *q is a lower bound of the set B of "bad" points in I*, contrary to $q > r = \operatorname{glb} B$. This contradiction shows that b must be "good," i.e.,

$$|f(b) - f(a)| \le g(b) - g(a) + [b - a + Q(b)]\varepsilon.$$

Now, letting $\varepsilon \to 0$, we obtain formula (1), and all is proved. □

Corollary 1. *If $f\colon E^1 \to E$ is relatively continuous and finite on $I = [a, b] \subseteq E^1$, and has a derivative on $I - Q$, then there is a real M such that*

$$|f(b) - f(a)| \le M(b - a) \text{ and } M \le \sup_{t \in I - Q} |f'(t)|. \tag{5}$$

Proof. Let

$$M_0 = \sup_{t \in I - Q} |f'(t)|.$$

If $M_0 < +\infty$, put $M = M_0 \ge |f'|$ on $I - Q$, and take $g(x) = Mx$ in Theorem 1. Then $g' = M \ge |f'|$ on $I - Q$, so formula (1) yields (5) since

$$g(b) - g(a) = Mb - Ma = M(b - a).$$

If, however, $M_0 = +\infty$, let

$$M = \left| \frac{f(b) - f(a)}{b - a} \right| < M_0.$$

Then (5) clearly is true. Thus the required M exists always.[3] □

Corollary 2. *Let f be as in Corollary 1. Then f is constant on I iff $f' = 0$ on $I - Q$.*

Proof. If $f' = 0$ on $I - Q$, then $M = 0$ in Corollary 1, so Corollary 1 yields, for any subinterval $[a, x]$ $(x \in I)$, $|f(x) - f(a)| \le 0$; i.e., $f(x) = f(a)$ for all $x \in I$. Thus f is constant on I.

Conversely, if so, then $f' = 0$, even on all of I. □

Corollary 3. *Let $f, g\colon E^1 \to E$ be relatively continuous and finite on $I = [a, b]$, and differentiable on $I - Q$. Then $f - g$ is constant on I iff $f' = g'$ on $I - Q$.*

Proof. Apply Corollary 2 to the function $f - g$. □

We can now also strengthen parts (ii) and (iii) of Corollary 4 in §2.

Theorem 2. *Let f be real and have the properties stated in Corollary 1. Then*

(i) *$f\uparrow$ on $I = [a, b]$ iff $f' \ge 0$ on $I - Q$; and*

(ii) *$f\downarrow$ on I iff $f' \le 0$ on $I - Q$.*

Proof. Let $f' \ge 0$ on $I - Q$. Fix any $x, y \in I$ $(x < y)$ and define $g(t) = 0$ on E^1. Then $|g'| = 0 \le f'$ on $I - Q$. Thus g and f satisfy Theorem 1 (with their roles *reversed*) on I, and certainly on the subinterval $[x, y]$. Thus we have

$$f(y) - f(x) \ge |g(y) - g(x)| = 0, \text{ i.e., } f(y) \ge f(x) \text{ whenever } y > x \text{ in } I,$$

so $f\uparrow$ on I.

[3] Note that M as defined here *depends on a and b*. So does M_0.

Conversely, if $f\uparrow$ on I, then for every $p \in I$, we must have $f'(p) \geq 0$, for otherwise, by Lemma 1 of §2, f would *decrease* at p. Thus $f' \geq 0$, even on *all* of I, and (i) is proved. Assertion (ii) is proved similarly. □

Problems on Complex and Vector-Valued Functions on E^1

1. Do the case $g'(r) = +\infty$ in Lemma 1.
[Hint: Show that there is $s > r$ with

$$g(x) - g(r) \geq (|f'(r)| + 1)(x - r) \geq |f(x) - f(r)| \text{ for } x \in (r,\, s).$$

Such x are "good."]

2. Do the case $r = p_n \in Q$ in Lemma 1.
[Hint: Show by continuity that there is $s > r$ such that $(\forall\, x \in (r,\, s))$

$$|f(x) - f(r)| < \frac{\varepsilon}{2^{n+1}} \text{ and } |g(x) - g(r)| < \frac{\varepsilon}{2^{n+1}}.$$

Show that all such x are "good" since $x > r = p_n$ implies

$$2^{-n} + Q(r) \leq Q(x). \quad \text{(Why?)}]$$

3. Show that Corollary 3 in §2 (hence also Theorem 2 in §2) fails for complex functions.
[Hint: Let $f(x) = e^{xi} = \cos x + i \cdot \sin x$. Verify that $|f'| = 1$ yet $f(2\pi) - f(0) = 0$.]

4. (i) Verify that all propositions of §4 hold also if f' and g' are only *right* derivatives on $I - Q$.

(ii) Do the same for *left* derivatives. (See footnote 2.)

5. (i) Prove that if $f\colon E^1 \to E$ is continuous and finite on $I = (a,\, b)$ and differentiable on $I - Q$, and if

$$\sup_{t \in I - Q} |f'(t)| < +\infty,$$

then f is uniformly continuous on I.

(ii) Moreover, if E is *complete* (e.g., $E = E^n$), then $f(a^+)$ and $f(b^-)$ exist and are finite.

[Hints: (i) Use Corollary 1. (ii) See the "hint" to Problem 11(iii) of Chapter 4, §8.]

6. Prove that if f is as in Theorem 2, with $f' \geq 0$ on $I - Q$ and $f' > 0$ at some $p \in I$, then $f(a) < f(b)$. Do it also with f' treated as a *right* derivative (see Problem 4).

7. Let $f,\, g\colon E^1 \to E^1$ be relatively continuous on $I = [a,\, b]$ and have right derivatives f'_+ and g'_+ (finite or infinite, but not *both* infinite) on $I - Q$.

(i) Prove that if

$$mg'_+ \leq f'_+ \leq Mg'_+ \text{ on } I - Q$$

for some fixed $m, M \in E^1$, then

$$m[g(b) - g(a)] \le f(b) - f(a) \le M[g(b) - g(a)].$$

[Hint: Apply Theorem 2 and Problem 4 to each of $Mg - f$ and $f - mg$.]

(ii) Hence prove that

$$m_0(b - a) \le f(b) - f(a) \le M_0(b - a),$$

where

$$m_0 = \inf f'_+[I - Q] \text{ and } M_0 = \sup f'_+[I - Q] \text{ in } E^*.$$

[Hint: Take $g(x) = x$ if $m_0 \in E^1$ or $M_0 \in E^1$. The *infinite* case is simple.]

8. (i) Let $f\colon (a, b) \to E$ be finite, continuous, with a right derivative on (a, b). Prove that $q = \lim_{x \to a^+} f'_+(x)$ exists (*finite*) iff

$$q = \lim_{x,\, y \to a^+} \frac{f(x) - f(y)}{x - y},$$

i.e., iff

$$(\forall\, \varepsilon > 0)\ (\exists\, c > a)\ (\forall\, x, y \in (a, c) \mid x \ne y) \quad \left|\frac{f(x) - f(y)}{x - y} - q\right| < \varepsilon.$$

[Hints: If so, let $y \to x^+$ (keeping x fixed) to obtain

$$(\forall\, x \in (a, c)) \quad |f'_+(x) - q| \le \varepsilon. \quad \text{(Why?)}$$

Conversely, if $\lim_{x \to a^+} f'_+(x) = q$, then

$$(\forall\, \varepsilon > 0)\ (\exists\, c > a)\ (\forall\, t \in (a, c)) \quad |f'_+(t) - q| < \varepsilon.$$

Put

$$M = \sup_{a < t < c} |f'_+(t) - q| \le \varepsilon \quad (\text{why } \le \varepsilon?)$$

and

$$h(t) = f(t) - tq, \quad t \in (a, b).$$

Apply Corollary 1 and Problem 4 to h on the interval $[x, y] \subseteq (a, c)$, to get

$$|f(y) - f(x) - (y - x)q| \le M(y - x) \le \varepsilon(y - x).$$

Proceed.]

(ii) Prove similar statements for the cases $q = \pm\infty$ and $x \to b^-$.
[Hint: In case $q = \pm\infty$, use Problem 7(ii) instead of Corollary 1.]

9. From Problem 8 deduce that if f is as indicated and if f'_+ is left continuous at some $p \in (a, b)$, then f also has a *left* derivative at p.

If f'_+ is also right continuous at p, then $f'_+(p) = f'_-(p) = f'(p)$.
[Hint: Apply Problem 8 to (a, p) and (p, b).]

10. In Problem 8, prove that if, in addition, E is complete and if

$$q = \lim_{x\to a^+} f'_+(x) \neq \pm\infty \quad \text{(finite)},$$

then $f(a^+) \neq \pm\infty$ exists, and

$$\lim_{x\to a^+} \frac{f(x) - f(a^+)}{x - a} = q;$$

similarly in case $\lim_{x\to b^-} f'_+(x) = r$.

If *both* exist, set $f(a) = f(a^+)$ and $f(b) = f(b^-)$. Show that then f becomes relatively continuous on $[a, b]$, with $f'_+(a) = q$ and $f'_-(b) = r$.
[Hint: If

$$\lim_{x\to a^+} f'_+(x) = q \neq \pm\infty,$$

then f'_+ is *bounded* on some subinterval (a, c), $a < c \leq b$ (why?), so f is uniformly continuous on (a, c), by Problem 5, and $f(a^+)$ exists. Let $y \to a^+$, as in the hint to Problem 8.]

11. Do Problem 9 in §2 for complex and vector-valued functions.
[Hint: Use Corollary 1 of §4.]

12. Continuing Problem 7, show that the *equalities*

$$m = \frac{f(b) - f(a)}{b - a} = M$$

hold iff f is *linear*, i.e., $f(x) = cx + d$ for some $c, d \in E^1$, and then $c = m = M$.

13. Let $f\colon E^1 \to C$ be as in Corollary 1, with $f \neq 0$ on I. Let g be the real part of f'/f.

(i) Prove that $|f|\!\uparrow$ on I iff $g \geq 0$ on $I - Q$.

(ii) Extend Problem 4 to this result.

14. Define $f\colon E^1 \to C$ by

$$f(x) = \begin{cases} x^2 e^{i/x} = x^2\left(\cos\frac{1}{x} + i \cdot \sin\frac{1}{x}\right) & \text{if } x > 0, \text{ and} \\ 0 & \text{if } x \leq 0. \end{cases}$$

Show that f is differentiable on $I = (-1, 1)$, yet $f'[I]$ is *not* a convex set in $E^2 = C$ (thus there is no analogue to Theorem 4 of §2).

§5. Antiderivatives (Primitives, Integrals)

Given $f\colon E^1 \to E$, we often have to find a function F such that $F' = f$ on I, or at least on $I - Q$.[1] We also require F to be relatively continuous and finite on I. This process is called *antidifferentiation* or *integration*.

Definition 1.

We call $F\colon E^1 \to E$ a *primitive*, or *antiderivative*, or an *indefinite integral*, of f on I iff

(i) F is relatively continuous and finite on I, and

(ii) F is differentiable, with $F' = f$, on $I - Q$ at least.

We then write

$$F = \int f, \text{ or } F(x) = \int f(x)\,dx, \text{ on } I.$$

(The latter is classical notation.)

If such an F exists (which is not always the case), we shall say that $\int f$ *exists on* I, or that f *has a primitive* (or *antiderivative*) on I, or that f is *primitively integrable* (briefly *integrable*) on I.

If $F' = f$ on a set $B \subseteq I$, we say that $\int f$ is *exact* on B and call F an *exact primitive on* B. Thus if $Q = \emptyset$, $\int f$ is exact on all of I.

Note 1. Clearly, if $F' = f$, then also $(F + c)' = f$ for a finite constant c. Thus the notation $F = \int f$ is rather incomplete; it means that F is *one of many* primitives. We now show that *all* of them have the form $F + c$ (or $\int f + c$).

Theorem 1. *If F and G are primitive to f on I, then $G - F$ is constant on I.*

Proof. By assumption, F and G are relatively continuous and finite on I; hence so is $G - F$. Also, $F' = f$ on $I - Q$ and $G' = f$ on $I - P$. (Q and P are countable, but possibly $Q \neq P$.)

Hence *both* F' and G' equal f *on* $I - S$, where $S = P \cup Q$, and S is countable itself by Theorem 2 of Chapter 1, §9.

Thus by Corollary 3 in §4, $F' = G'$ on $I - S$ implies $G - F = c$ (constant) on each $[x, y] \subseteq I$; hence $G - F = c$ (or $G = F + c$) on I. □

Definition 2.

If $F = \int f$ on I and if $a, b \in I$ (where $a \le b$ or $b \le a$), we define

$$\int_a^b f = \int_a^b f(x)\,dx = F(b) - F(a), \text{ also written } F(x)\Big|_a^b. \tag{1}$$

[1] In this section, Q, P, and S shall denote *countable* sets, F', G', and H' are *finite derivatives*, and I is a finite or infinite nondegenerate interval in E^1.

This expression is called the *definite integral* of f *from* a *to* b.[2]

The definite integral of f from a to b is independent of the particular choice of the primitive F for f, and thus *unambiguous*, for if G is another primitive, Theorem 1 yields $G = F + c$, so

$$G(b) - G(a) = F(b) + c - [F(a) + c] = F(b) - F(a),$$

and it does not matter whether we take F or G.

Note that $\int_a^b f(x)\,dx$, or $\int_a^b f$, is a *constant* in the range space E (a *vector* if f is vector valued). The "x" in $\int_a^b f(x)\,dx$ is a "dummy variable" only, and it may be replaced by any other letter. Thus

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = F(b) - F(a).$$

On the other hand, the *indefinite* integral is a *function*: $F\colon E^1 \to E$.

Note 2. We may, however, *vary* a or b (or both) in (1). Thus, keeping a fixed and varying b, we can define a function

$$G(t) = \int_a^t f = F(t) - F(a), \quad t \in I.$$

Then $G' = F' = f$ on I, and $G(a) = F(a) - F(a) = 0$. Thus if $\int f$ exists on I, *f has a (unique) primitive G on I such that $G(a) = 0$.* (It is unique by Theorem 1. Why?)

Examples.

(a) Let

$$f(x) = \frac{1}{x} \text{ and } F(x) = \ln|x|, \text{ with } F(0) = f(0) = 0.$$

Then $F' = f$ and $F = \int f$ on $(-\infty, 0)$ and on $(0, +\infty)$ but *not on* E^1, since F is discontinuous at 0, contrary to Definition 1. We compute

$$\int_1^2 f = \ln 2 - \ln 1 = \ln 2.$$

(b) On E^1, let

$$f(x) = \frac{|x|}{x} \text{ and } F(x) = |x|, \text{ with } f(0) = 1.$$

Here F *is* continuous and $F' = f$ on $E^1 - \{0\}$. Thus $F = \int f$ on E^1, exact on $E^1 - \{0\}$. Here $I = E^1$, $Q = \{0\}$.

[2] The numbers a and b are called the *bounds* of the integral.

We compute

$$\int_{-2}^{2} f = F(2) - F(-2) = 2 - 2 = 0$$

(*even though* f *never vanishes on* E^1).

Basic properties of integrals follow from those of derivatives. Thus we have the following.

Corollary 1 (linearity). *If* $\int f$ *and* $\int g$ *exist on* I, *so does* $\int(pf+qg)$ *for any scalars* p, q (*in the scalar field of* E).[3] *Moreover, for any* $a, b \in I$, *we obtain*

(i) $\displaystyle\int_a^b (pf+qg) = p\int_a^b f + q\int_a^b g;$

(ii) $\displaystyle\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g;$ *and*

(iii) $\displaystyle\int_a^b pf = p\int_a^b f.$

Proof. By assumption, there are F and G such that

$$F' = f \text{ on } I - Q \text{ and } G' = g \text{ on } I - P.$$

Thus, setting $S = P \cup Q$ and $H = pF + qG$, we have

$$H' = pF' + qG' = pf + qg \text{ on } I - S,$$

with P, Q, and S countable. Also, $H = pF + qG$ is relatively continuous and finite on I, as are F and G.

Thus by definition, $H = \int(pf+qg)$ exists on I, and by (1),

$$\int_a^b (pf+qg) = H(b) - H(a) = pF(b) + qG(b) - pF(a) - qG(a) = p\int_a^b f + q\int_a^b g,$$

proving (i*).

With $p = 1$ and $q = \pm 1$, we obtain (ii*).

Taking $q = 0$, we get (iii*). □

Corollary 2. *If both* $\int f$ *and* $\int |f|$ *exist on* $I = [a, b]$, *then*

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

[3] In the case $f, g\colon E^1 \to E^*$ (C), we assume $p, q \in E^1$ (C). If f and g are scalar valued, we also allow p and q to be *vectors* in E.

Proof. As before, let

$$F' = f \text{ and } G' = |f| \text{ on } I - S \ (S = Q \cup P, \text{ all countable}),$$

where F and G are relatively continuous and finite on I and $G = \int |f|$ is *real*. Also, $|F'| = |f| = G'$ on $I - S$. Thus by Theorem 1 of §4,

$$|F(b) - F(a)| \leq G(b) - G(a) = \int_a^b |f|. \quad \square$$

Corollary 3. *If $\int f$ exists on $I = [a, b]$, exact on $I - Q$, then*

$$\left|\int_a^b f\right| \leq M(b - a)$$

for some real

$$M \leq \sup_{t \in I - Q} |f(t)|.$$

This is simply Corollary 1 of §4, when applied to a primitive, $F = \int f$.

Corollary 4. *If $F = \int f$ on I and $f = g$ on $I - Q$, then F is also a primitive of g, and*

$$\int_a^b f = \int_a^b g \quad \textit{for } a, b \in I.$$

(*Thus we may arbitrarily redefine f on a countable Q.*)

Proof. Let $F' = f$ on $I - P$. Then $F' = g$ on $I - (P \cup Q)$. The rest is clear. $\square$

Corollary 5 (integration by parts)**.** *Let f and g be real or complex* (*or let f be scalar valued and g vector valued*), *both relatively continuous on I and differentiable on $I - Q$. Then if $\int f'g$ exists on I, so does $\int fg'$, and we have*

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g \quad \textit{for any } a, b \in I. \tag{2}$$

Proof. By assumption, fg is relatively continuous and finite on I, and

$$(fg)' = fg' + f'g \text{ on } I - Q.$$

Thus, setting $H = fg$, we have $H = \int (fg' + f'g)$ on I. Hence by Corollary 1, if $\int f'g$ exists on I, so does $\int ((fg' + f'g) - f'g) = \int fg'$, and

$$\int_a^b fg' + \int_a^b f'g = \int_a^b (fg' + f'g) = H(b) - H(a) = f(b)g(b) - f(a)g(a).$$

Thus (2) follows. $\square$

The proof of the next three corollaries is left to the reader.

Corollary 6 (additivity of the integral)**.** *If $\int f$ exists on I then, for a, b, $c \in I$, we have*

(i) $\displaystyle\int_a^b f = \int_a^c f + \int_c^b f;$

(ii) $\displaystyle\int_a^a f = 0;$ *and*

(iii) $\displaystyle\int_b^a f = -\int_a^b f.$

Corollary 7 (componentwise integration)**.** *A function $f\colon E^1 \to E^n$ (C^n) is integrable on I iff all its components $(f_1, f_2, \ldots, f_n)$ are, and then (by Theorem 5 in* §1*)*

$$\int_a^b f = \Big(\int_a^b f_1, \ldots, \int_a^b f_n\Big) = \sum_{k=1}^n \vec{e}_k \int_a^b f_k \quad \textit{for any } a, b \in I.$$

Hence if f is complex,

$$\int_a^b f = \int_a^b f_{\text{re}} + i \cdot \int_a^b f_{\text{im}}$$

(*see Chapter* 4, §3, *Note* 5).

Examples (continued)**.**

(c) Define $f\colon E^1 \to E^3$ by

$$f(x) = (a \cdot \cos x,\ a \cdot \sin x,\ 2cx), \quad a, c \in E^1.$$

Verify that

$$\int_0^\pi f(x)\,dx = (a \cdot \sin x, -a \cdot \cos x, cx^2)\Big|_0^\pi = (0, 2a, c\pi^2) = 2a\vec{j} + c\pi^2\vec{k}.$$

(d) $\displaystyle\int_0^\pi e^{ix}\,dx = \int_0^\pi (\cos x + i \cdot \sin x)\,dx = (\sin x - i \cdot \cos x)\Big|_0^\pi = 2i.$

Corollary 8. *If $f = 0$ on $I - Q$, then $\int f$ exists on I, and*

$$\Big|\int_a^b f\Big| = \int_a^b |f| = 0 \quad \textit{for } a, b \in I.$$

Theorem 2 (change of variables)**.** *Suppose $g\colon E^1 \to E^1$ (real) is differentiable on I, while $f\colon E^1 \to E$ has a primitive on $g[I]$,*[4] *exact on $g[I - Q]$.*

[4] Note that $g[I]$ is an *interval*, for g has the Darboux property (Chapter 4, §9, Note 1).

Then

$$\int f\big(g(x)\big)g'(x)\,dx \quad \left(i.e., \int (f\circ g)g'\right)$$

exists on I, *and for any* $a, b \in I$, *we have*

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_p^q f(y)\,dy, \text{ where } p = g(a) \text{ and } q = g(b). \qquad (3)$$

Thus, using classical notation, we may substitute $y = g(x)$, *provided that we also substitute* $dy = g'(x)\,dx$ *and change the bounds of integrals* (3). Here we treat the expressions dy and $g'(x)\,dx$ purely formally, without assigning them any separate meaning outside the context of the integrals.

Proof. Let $F = \int f$ on $g[I]$, and $F' = f$ on $g[I - Q]$. Then the composite function $H = F \circ g$ is relatively continuous and finite on I. (Why?) By Theorem 3 of §1,

$$H'(x) = F'\big(g(x)\big)g'(x) \text{ for } x \in I - Q;$$

i.e.,

$$H' = (F' \circ g)g' \text{ on } I - Q.$$

Thus $H = \int (f \circ g)g'$ exists on I, and

$$\int_a^b (f\circ g)g' = H(b) - H(a) = F\big(g(b)\big) - F\big(g(a)\big) = F(q) - F(p) = \int_p^q f. \quad \square$$

Note 3. The theorem does not require that g be *one to one* on I, but if it is, then one can drop the assumption that $\int f$ is *exact on* $g[I - Q]$. (See Problem 4.)

Examples (continued)**.**

(e) Find $\displaystyle\int_0^{\pi/2} \sin^2 x \cdot \cos x\,dx$.

Here $f(y) = y^2$, $y = g(x) = \sin x$, $dy = \cos x\,dx$, $F(y) = y^3/3$, $a = 0$, $b = \pi/2$, $p = \sin 0 = 0$, and $q = \sin(\pi/2) = 1$, so (3) yields

$$\int_0^{\pi/2} \sin^2 x \cdot \cos x\,dx = \int_0^1 y^2\,dy = \frac{y^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

For *real* functions, we obtain some inferences dealing with *inequalities*.

Theorem 3. *If* $f, g\colon E^1 \to E^1$ *are integrable on* $I = [a, b]$, *then we have the following*:

(i) $f \geq 0$ *on* $I - Q$ *implies* $\int_a^b f \geq 0$.

(i′) $f \leq 0$ *on* $I - Q$ *implies* $\int_a^b f \leq 0$.

(ii) *$f \geq g$ on $I - Q$ implies*

$$\int_a^b f \geq \int_a^b g \text{ (dominance law).}$$

(iii) *If $f \geq 0$ on $I - Q$ and $a \leq c \leq d \leq b$, then*

$$\int_a^b f \geq \int_c^d f \text{ (monotonicity law).}$$

(iv) *If $\int_a^b f = 0$, and $f \geq 0$ on $I - Q$, then $f = 0$ on some $I - P$, P countable.*

Proof. By Corollary 4, we may redefine f on Q so that our assumptions in (i)–(iv) hold on *all* of I. Thus we write "I" for "$I - Q$."

By assumption, $F = \int f$ and $G = \int g$ exist on I. Here F and G are relatively continuous and finite on $I = [a, b]$, with $F' = f$ and $G' = g$ on $I - P$, for another countable set P (this P *cannot* be omitted). Now consider the cases (i)–(iv). (P is fixed henceforth.)

(i) Let $f \geq 0$ on I; i.e., $F' = f \geq 0$ on $I - P$. Then by Theorem 2 in §4, $F\uparrow$ on $I = [a, b]$. Hence $F(a) \leq F(b)$, and so

$$\int_a^b f = F(b) - F(a) \geq 0.$$

One proves (i′) similarly.

(ii) If $f - g \geq 0$, then by (i),

$$\int_a^b (f - g) = \int_a^b f - \int_a^b g \geq 0,$$

so $\int_a^b f \geq \int_a^b g$, as claimed.

(iii) Let $f \geq 0$ on I and $a \leq c \leq d \leq b$. Then by (i),

$$\int_a^c f \geq 0 \text{ and } \int_d^b f \geq 0.$$

Thus by Corollary 6,

$$\int_a^b f = \int_a^c f + \int_c^d f + \int_d^b f \geq \int_c^d f,$$

as asserted.

(iv) Seeking a contradiction, suppose $\int_a^b f = 0$, $f \geq 0$ on I, yet $f(p) > 0$ for some $p \in I - P$ (P as above), so $F'(p) = f(p) > 0$.

Now if $a \le p < b$, Lemma 1 of §2 yields $F(c) > F(p)$ for some $c \in (p,\, b]$. Then by (iii),

$$\int_a^b f \ge \int_p^c f = F(c) - F(p) > 0,$$

contrary to $\int_a^b f = 0$; similarly in case $a < p \le b$. □

Note 4. Hence

$$\int_a^b |f| = 0 \text{ implies } f = 0 \text{ on } [a, b] - P$$

(P countable), even for vector-valued functions (for $|f|$ is always *real*, and so Theorem 3 applies).

However, $\int_a^b f = 0$ *does not suffice*, even for real functions (unless f is *sign-constant*). For example,

$$\int_0^{2\pi} \sin x\, dx = 0, \text{ yet } \sin x \not\equiv 0 \text{ on any } I - P.$$

See also Example (b).

Corollary 9 (first law of the mean)**.** *If f is real and $\int f$ exists on $[a,\, b]$, exact on $(a,\, b)$, then*

$$\int_a^b f = f(q)(b-a) \textit{ for some } q \in (a,\, b).$$

Proof. Apply Corollary 3 in §2 to the function $F = \int f$. □

Caution: Corollary 9 may fail if $\int f$ is inexact at some $p \in (a,\, b)$. (Exactness on $[a, b] - Q$ *does not* suffice, as it does not in Corollary 3 of §2, used here.) Thus in Example (b) above, $\int_{-2}^{2} f = 0$. Yet for no q is $f(q)(2+2) = 0$, since $f(q) = \pm 1$. The reason is that $\int f$ is inexact *just at* 0, an interior point of $[-2,\, 2]$.

Problems on Antiderivatives

1. Prove in detail Corollaries 3, 4, 6, 7, 8, and 9 and Theorem 3(i′) and (iv).

2. In Examples (a) and (b) discuss continuity and differentiability of f and F at 0. In (a) show that $\int f$ *does not exist on any interval* $(-a,\, a)$.
[Hint: Use Theorem 1.]

3. Show that Theorem 2 holds also if g is relatively continuous on I and differentiable on $I - Q$.

4. Under the assumptions of Theorem 2, show that if g is one to one on I, then *automatically* $\int f$ is exact on $g[I - Q]$ (Q countable).
[Hint: If $F = \int f$ on $g[I]$, then

$$F' = f \text{ on } g[I] - P,\ P \text{ countable.}$$

Let $Q = g^{-1}[P]$. Use Problem 6 of Chapter 1, §§4–7 and Problem 2 of Chapter 1, §9 to show that Q is countable and $g[I] - P = g[I - Q]$.]

5. Prove Corollary 5 for *dot products* $f \cdot g$ of vector-valued functions.

6. Prove that if $\int f$ exists on $[a, p]$ and $[p, b]$, then it exists on $[a, b]$. By induction, extend this to unions of n adjacent intervals.
[Hint: Choose $F = \int f$ on $[a, p]$ and $G = \int f$ on $[p, b]$ such that $F(p) = G(p)$. (Why do such F, G exist?) Then construct a primitive $H = \int f$ that is relatively continuous on *all* of $[a, b]$.]

7. Prove the *weighted law of the mean*: If g is real and nonnegative on $I = [a, b]$, and if $\int g$ and $\int gf$ exist on I for some $f \colon E^1 \to E$, then there is a finite $c \in E$ with

$$\int_a^b gf = c \int_a^b g.$$

(The value c is called a *g-weighted mean* of f.)
[Hint: If $\int_a^b g > 0$, put

$$c = \int_a^b gf \Big/ \int_a^b g.$$

If $\int_a^b g = 0$, use Theorem 3(i) and (iv) to show that also $\int_a^b gf = 0$, so *any* c will do.]

8. In Problem 7, prove that if, in addition, f *is real* and has the Darboux property on I, then $c = f(q)$ for some $q \in I$ (the *second law of the mean*).
[Hint: Choose c as in Problem 7. If $\int_a^b g > 0$, put

$$m = \inf f[I] \text{ and } M = \sup f[I], \text{ in } E^*,$$

so $m \leq f \leq M$ on I. Deduce that

$$m \int_a^b g \leq \int_a^b gf \leq M \int_a^b g,$$

whence $m \leq c \leq M$.

If $m < c < M$, then $f(x) < c < f(y)$ for some $x, y \in I$ (why?), so the Darboux property applies.

If $c = m$, then $g \cdot (f - c) \geq 0$ and Theorem 3(iv) yields $gf = gc$ on $I - P$. (Why?) Deduce that $f(q) = c$ if $g(q) \neq 0$ and $q \in I - P$. (Why does such a q exist?)

What if $c = M$?]

9. Taking $g(x) \equiv 1$ in Problem 8, obtain a new version of Corollary 9. State it precisely!

⇒**10.** Prove that if $F = \int f$ on $I = (a, b)$ and f is right (left) continuous and finite at $p \in I$, then

$$f(p) = F'_+(p) \text{ (respectively, } F'_-(p)).$$

Deduce that *if f is continuous and finite on I, all its primitives on I are exact on I.*

[Hint: Fix $\varepsilon > 0$. If f is right continuous at p, there is $c \in I$ $(c > p)$, with

$$|f(x) - f(p)| < \varepsilon \text{ for } x \in [p, c).$$

Fix such an x. Let

$$G(t) = F(t) - tf(p), \quad t \in E^1.$$

Deduce that $G'(t) = f(t) - f(p)$ for $t \in I - Q$.

By Corollary 1 of §4,

$$|G(x) - G(p)| = |F(x) - F(p) - (x - p)f(p)| \le M(x - p),$$

with $M \le \varepsilon$. (Why?) Hence

$$\left|\frac{\Delta F}{\Delta x} - f(p)\right| \le \varepsilon \text{ for } x \in [p, c),$$

and so

$$\lim_{x \to p^+} \frac{\Delta F}{\Delta x} = f(p) \quad \text{(why?)};$$

similarly for a *left*-continuous f.]

11. State and solve Problem 10 for the case $I = [a, b]$.

12. (i) Prove that if f is constant $(f = c \neq \pm\infty)$ on $I - Q$, then

$$\int_a^b f = (b - a)c \quad \text{for } a, b \in I.$$

(ii) Hence prove that if $f = c_k \neq \pm\infty$ on

$$I_k = [a_k, a_{k+1}), \quad a = a_0 < a_1 < \cdots < a_n = b,$$

then $\int f$ exists on $[a, b]$, and

$$\int_a^b f = \sum_{k=0}^{n-1} (a_{k+1} - a_k) c_k.$$

Show that this is true also if $f = c_k \neq \pm\infty$ *on* $I_k - Q_k$.

[Hint: Use Problem 6.]

13. Prove that if $\int f$ exists on each $I_n = [a_n, b_n]$, where

$$a_{n+1} \le a_n \le b_n \le b_{n+1}, \quad n = 1, 2, \ldots,$$

then $\int f$ exists on

$$I = \bigcup_{n=1}^{\infty} [a_n,\, b_n],$$

itself an interval with endpoints $a = \inf a_n$ and $b = \sup b_n$, $a,\, b \in E^*$.
[Hint: Fix some $c \in I_1$. Define

$$H_n(t) = \int_c^t f \text{ on } I_n,\ n = 1,\, 2,\, \ldots .$$

Prove that

$$(\forall\, n \leq m) \quad H_n = H_m \text{ on } I_n \text{ (since } \{I_n\}\uparrow).$$

Thus $H_n(t)$ is *the same* for *all* n such that $t \in I_n$, so we may simply write H for H_n on $I = \bigcup_{n=1}^{\infty} I_n$. Show that $H = \int f$ on all of I; verify that I is, indeed, an *interval*.]

14. Continuing Problem 13, prove that $\int f$ exists on an interval I iff it exists on each closed subinterval $[a,\, b] \subseteq I$.
[Hint: Show that *each* I is the union of an *expanding* sequence $I_n = [a_n,\, b_n]$. For example, if $I = (a,\, b)$, $a,\, b \in E^1$, put

$$a_n = a + \frac{1}{n} \text{ and } b_n = b - \frac{1}{n} \text{ for large } n \text{ (how large?)},$$

and show that

$$I = \bigcup_n [a_n,\, b_n] \text{ over such } n.]$$

§6. Differentials. Taylor's Theorem and Taylor's Series

Recall (Theorem 2 of §1) that a function f is differentiable at p iff

$$\Delta f = f'(p)\Delta x + \delta(x)\Delta x,$$

with $\lim_{x\to p} \delta(x) = \delta(p) = 0$. It is customary to write df for $f'(p)\Delta x$ and $o(\Delta x)$ for $\delta(x)\Delta x$;[1] df is called the *differential* of f (at p and x). Thus

$$\Delta f = df + o(\Delta x);$$

i.e., df approximates Δf to within $o(\Delta x)$.

More generally, given any function $f\colon E^1 \to E$ and $p, x \in E^1$, we define

$$d^n f = d^n f(p,\, x) = f^{(n)}(p)(x - p)^n, \quad n = 0,\, 1,\, 2,\, \ldots, \tag{1}$$

[1] This is the so-called "little o" notation. Given $g\colon E^1 \to E^1$, we write $o(g(x))$ for any expression of the form $\delta(x)g(x)$, with $\delta(x) \to 0$. In our case, $g(x) = \Delta x$.

where $f^{(n)}$ is the nth derived function (Definition 2 in §1); $d^n f$ is called the *nth differential, or differential of order n*, of f (at p and x). In particular, $d^1 f = f'(p)\Delta x = df$.[2] By our conventions, $d^n f$ is *always* defined, as is $f^{(n)}$.

As we shall see, good approximations of Δf (suggested by Taylor) can often be obtained by using *higher* differentials (1), as follows:

$$\Delta f = df + \frac{d^2 f}{2!} + \frac{d^3 f}{3!} + \cdots + \frac{d^n f}{n!} + R_n, \quad n = 1, 2, 3, \ldots, \tag{2}$$

where

$$R_n = \Delta f - \sum_{k=1}^{n} \frac{d^k f}{k!} \quad \text{(the "\textit{remainder term}")}$$

is the error of the approximation. Substituting the values of Δf and $d^k f$ and transposing $f(p)$, we have

$$f(x) = f(p) + \frac{f'(p)}{1!}(x-p) + \frac{f''(p)}{2!}(x-p)^2 + \cdots + \frac{f^{(n)}(p)}{n!}(x-p)^n + R_n. \tag{3}$$

Formula (3) is known as the *nth Taylor expansion* of f about p (with remainder term R_n to be estimated). Usually we treat p as fixed and x as variable. Writing $R_n(x)$ for R_n and setting

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(p)}{k!}(x-p)^k,$$

we have

$$f(x) = P_n(x) + R_n(x).$$

The function $P_n \colon E^1 \to E$ so defined is called the *nth Taylor polynomial* for f about p. Thus (3) yields *approximations of f by polynomials P_n*, $n = 1, 2, 3, \ldots$. This is one way of interpreting it. The other (easy to remember) one is (2), which gives *approximations of Δf by the $d^k f$*. It remains, however, to find a good estimate for R_n. We do it next.

Theorem 1 (Taylor)**.** *Let the function $f \colon E^1 \to E$ and its first n derived functions be relatively continuous and finite on an interval I and differentiable on $I - Q$ (Q countable). Let $p, x \in I$. Then formulas (2) and (3) hold, with*

$$R_n = \frac{1}{n!}\int_p^x f^{(n+1)}(t) \cdot (x-t)^n \, dt \quad \text{("\textit{integral form of } } R_n\text{")} \tag{3'}$$

and

$$|R_n| \le M_n \frac{|x-p|^{n+1}}{(n+1)!} \text{ \textit{for some real} } M_n \le \sup_{t \in I-Q} |f^{(n+1)}(t)|. \tag{3''}$$

[2] Footnote 2 of §1 applies to $d^n f$, as it does to Δf (and to R_n defined below).

Proof. By definition, $R_n = f - P_n$, or

$$R_n = f(x) - f(p) - \sum_{k=1}^{n} f^{(k)}(p)\frac{(x-p)^k}{k!}.$$

We use the right side as a "pattern" to define a function $h\colon E^1 \to E$. This time, we keep x fixed (say, $x = a \in I$) and replace p by a variable t. Thus we set

$$h(t) = f(a) - f(t) - \frac{f'(t)}{1!}(a-t) - \cdots - \frac{f^{(n)}(t)}{n!}(a-t)^n \text{ for all } t \in E^1. \quad (4)$$

Then $h(p) = R_n$ and $h(a) = 0$. Our assumptions imply that h is relatively continuous and finite on I, and differentiable on $I - Q$. Differentiating (4), we see that *all cancels out except for one term*

$$h'(t) = -f^{(n+1)}(t)\frac{(a-t)^n}{n!}, \quad t \in I - Q. \quad \text{(Verify!)} \quad (5)$$

Hence by Definitions 1 and 2 of §5,

$$-h(t) = \frac{1}{n!}\int_t^a f^{(n+1)}(s)(a-s)^n\,ds \quad \text{on } I$$

and

$$\frac{1}{n!}\int_p^a f^{(n+1)}(t)(a-t)^n\,dt = -h(a) + h(p) = 0 + R_n = R_n \quad (\text{for } h(p) = R_n).$$

As $x = a$, (3′) is proved.

Next, let

$$M = \sup_{t \in I - Q} |f^{(n+1)}(t)|.$$

If $M < +\infty$, define

$$g(t) = M\frac{(t-a)^{n+1}}{(n+1)!} \text{ for } t \geq a \text{ and } g(t) = -M\frac{(a-t)^{n+1}}{(n+1)!} \text{ for } t \leq a.$$

In both cases,

$$g'(t) = M\frac{|a-t|^n}{n!} \geq |h'(t)| \text{ on } I - Q \text{ by (5).}$$

Hence, applying Theorem 1 in §4 to the functions h and g on the interval $[a, p]$ (or $[p, a]$), we get

$$|h(p) - h(a)| \leq |g(p) - g(a)|,$$

or

$$|R_n - 0| \leq M\frac{|a-p|^{n+1}}{(n+1)!}.$$

Thus (3″) follows, with $M_n = M$.

Finally, if $M = +\infty$, we put

$$M_n = |R_n| \frac{(n+1)!}{|a-p|^{n+1}} < M. \quad \square$$

For *real* functions, we obtain some additional estimates of R_n.

Theorem 1′. *If f is real and $n+1$ times differentiable on I, then for $p \neq x$ $(p,\, x \in I)$, there are q_n, q_n' in the interval $(p,\, x)$ (respectively, $(x,\, p)$) such that*

$$R_n = \frac{f^{(n+1)}(q_n)}{(n+1)!}(x-p)^{n+1} \tag{5′}$$

and

$$R_n = \frac{f^{(n+1)}(q_n')}{n!}(x-p)(x-q_n')^n. \tag{5″}$$

(Formulas (5′) and (5″) are known as the *Lagrange* and *Cauchy forms of* R_n, respectively.)

Proof. Exactly as in the proof of Theorem 1, we obtain the function h and formula (5). By our present assumptions, h is differentiable (hence continuous) on I, so we may apply to it Cauchy's law of the mean (Theorem 2 of §2) on the interval $[a,\, p]$ (or $[p,\, a]$ if $p < a$), where $a = x \in I$.

For this purpose, we shall associate h with another suitable function g (to be specified later). Then by Theorem 2 of §2, there is a real $q \in (a,\, p)$ (respectively, $q \in (p,\, a)$) such that

$$g'(q)[h(a) - h(p)] = h'(q)[g(a) - g(p)].$$

Here by the previous proof, $h(a) = 0$, $h(p) = R_n$, and

$$h'(q) = -\frac{f^{(n+1)}}{n!}(a-q)^n.$$

Thus

$$g'(q) \cdot R_n = \frac{f^{(n+1)}(q)}{n!}(a-q)^n[g(a) - g(p)]. \tag{6}$$

Now define g by

$$g(t) = a - t, \quad t \in E^1.$$

Then

$$g(a) - g(p) = -(a-p) \text{ and } g'(q) = -1,$$

so (6) yields (5″) (with $q_n' = q$ and $a = x$).

Similarly, setting $g(t) = (a-t)^{n+1}$, we obtain (5′). (Verify!) Thus all is proved. $\square$

Note 1. In ($5'$) and ($5''$), the numbers q_n and q'_n *depend on* n and are *different* in general ($q_n \neq q'_n$), for they depend on the choice of the function g. Since they are between p and x, they may be written as

$$q_n = p + \theta_n(x-p) \text{ and } q'_n = p + \theta'_n(x-p),$$

where $0 < \theta_n < 1$ and $0 < \theta'_n < 1$. (Explain!)

Note 2. For any function $f\colon E^1 \to E$, the Taylor polynomials P_n are partial sums of a power series, called the *Taylor series for* f (about p). We say that f *admits* such a series on a set B iff the series converges to f on B; i.e.,

$$f(x) = \lim_{n\to\infty} P_n(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(p)}{n!}(x-p)^n \neq \pm\infty \text{ for } x \in B. \tag{7}$$

This is clearly the case iff

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} [f(x) - P_n(x)] = 0 \text{ for } x \in B;$$

briefly, $R_n \to 0$. Thus

$$f \text{ admits a Taylor series (about } p) \text{ iff } R_n \to 0.$$

Caution: The convergence of the series *alone* (be it pointwise or uniform) does not suffice. Sometimes the series converges to a sum *other than* $f(x)$; then (7) *fails*. Thus all depends on the *necessary and sufficient condition*: $R_n \to 0$.

Before giving examples, we introduce a convenient notation.

Definition 1.

We say that f is *of class* CD^n, or *continuously differentiable* n *times*, on a set B iff f is n times differentiable on B, and $f^{(n)}$ is relatively continuous on B. Notation: $f \in \mathrm{CD}^n$ (on B).

If this holds for *each* $n \in N$, we say that f is *infinitely differentiable* on B and write $f \in \mathrm{CD}^\infty$ (on B).

The notation $f \in \mathrm{CD}^0$ means that f is finite and relatively continuous (all on B).

Examples.

(a) Let

$$f(x) = e^x \text{ on } E^1.$$

Then

$$(\forall\, n) \quad f^{(n)}(x) = e^x,$$

so $f \in \mathrm{CD}^\infty$ on E^1. At $p = 0$, $f^{(n)}(p) = 1$, so we obtain by Theorem $1'$

(using (5′) and Note 1)

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^{\theta_n x}}{(n+1)!}\,x^{n+1}, \quad 0 < \theta_n < 1. \tag{8}$$

Thus on an interval $[-a, a]$,

$$e^x \approx 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

to within an error R_n (> 0 if $x > 0$) with

$$|R_n| < e^a \frac{a^{n+1}}{(n+1)!},$$

which tends to 0 as $n \to +\infty$. For $a = 1 = x$, we get

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n \text{ with } 0 < R_n < \frac{e^1}{(n+1)!}\,. \tag{9}$$

Taking $n = 10$, we have

$$e \approx 2.7182818|011463845\ldots$$

with a nonnegative error of no more than

$$\frac{e}{11!} = 0.00000006809869\ldots;$$

all digits are correct before the vertical bar.

(b) Let

$$f(x) = e^{-1/x^2} \text{ with } f(0) = 0.$$

As $\lim_{x\to 0} f(x) = 0 = f(0)$, f is continuous at 0.[3] We now show that $f \in \mathrm{CD}^\infty$ on E^1.

For $x \neq 0$, this is clear; moreover, induction yields

$$f^{(n)}(x) = e^{-1/x^2} x^{-3n} S_n(x),$$

where S_n is a polynomial in x of degree $2(n-1)$ (this is all we need know about S_n). A repeated application of L'Hôpital's rule then shows that

$$\lim_{x\to 0} f^{(n)}(x) = 0 \text{ for each } n.$$

To find $f'(0)$, we have to use the definition of a derivative:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0},$$

[3] At other points, f is continuous by the continuity of exponentials.

or by L'Hôpital's rule,

$$f'(0) = \lim_{x \to 0} \frac{f'(x)}{1} = 0.$$

Using induction again, we get

$$f^{(n)}(0) = 0, \quad n = 1, 2, \ldots.$$

Thus, indeed, f has finite derivatives of all orders at each $x \in E^1$, including $x = 0$, so $f \in \mathrm{CD}^\infty$ on E^1, as claimed.

Nevertheless, any attempt to use formula (3) *at* $p = 0$ yields nothing. As all $f^{(n)}$ vanish at 0, so do all terms except R_n. Thus no approximation by polynomials results—we only get $P_n = 0$ on E^1 and $R_n(x) = e^{-1/x^2}$. R_n does not tend to 0 except at $x = 0$, so f *admits no Taylor series about* 0 (except on $E = \{0\}$).[4]

Taylor's theorem also yields *sufficient* conditions for maxima and minima, as we see in the following theorem.

Theorem 2. *Let* $f\colon E^1 \to E^*$ *be of class* CD^n *on* $G_p(\delta)$ *for an* even *number* $n \geq 2$, *and let*

$$f^{(k)}(p) = 0 \text{ for } k = 1, 2, \ldots, n-1,$$

while

$$f^{(n)}(p) < 0 \text{ (respectively, } f^{(n)}(p) > 0).$$

Then $f(p)$ *is the maximum* (*respectively*, *minimum*) *value of* f *on some* $G_p(\varepsilon)$, $\varepsilon \leq \delta$.

If, *however*, *these conditions hold for some* odd $n \geq 1$ (*i.e.*, *the first nonvanishing derivative at* p *is of odd order*), f *has no maximum or minimum at* p.

Proof. As

$$f^{(k)}(p) = 0, \quad k = 1, 2, \ldots, n-1,$$

Theorem 1′ (with n replaced by $n-1$) yields

$$f(x) = f(p) + f^{(n)}(q_n)\frac{(x-p)^n}{n!} \quad \text{for all } x \in G_p(\delta),$$

with q_n between x and p.

Also, as $f \in \mathrm{CD}^n$, $f^{(n)}$ is continuous at p. Thus if $f^{(n)}(p) < 0$, then $f^{(n)} < 0$ on some $G_p(\varepsilon)$, $0 < \varepsilon \leq \delta$. However, $x \in G_p(\varepsilon)$ implies $q_n \in G_p(\varepsilon)$, so

$$f^{(n)}(q_n) < 0,$$

[4] Taylor's series with $p = 0$ is often called the *Maclaurin series* (though without proper justification). As we see, it may fail even if $f \in \mathrm{CD}^\infty$ near 0.

while

$$(x-p)^n \geq 0 \text{ if } n \text{ is } even.$$

It follows that

$$f^{(n)}(q_n)\frac{(x-p)^n}{n!} \leq 0,$$

and so

$$f(x) = f(p) + f^{(n)}(q_n)\frac{(x-p)^n}{n!} \leq f(p) \quad \text{for } x \in G_p(\varepsilon),$$

i.e., $f(p)$ is the *maximum* value of f on $G_p(\varepsilon)$, as claimed.

Similarly, in the case $f^{(n)}(p) > 0$, a *minimum* would result.

If, however, n is *odd*, then $(x-p)^n$ is negative for $x < p$ but positive for $x > p$. The same argument then shows that $f(x) < f(p)$ on one side of p and $f(x) > f(p)$ on the other side; thus no local maximum or minimum can exist at p. This completes the proof. □

Examples.

(a′) Let

$$f(x) = x^2 \text{ on } E^1 \text{ and } p = 0.$$

Then

$$f'(x) = 2x \text{ and } f''(x) = 2 > 0,$$

so

$$f'(0) = 0 \text{ and } f''(0) = 2 > 0.$$

By Theorem 2, $f(p) = 0^2 = 0$ is a minimum value.

It turns out to be a minimum *on all of* E^1. Indeed, $f'(x) > 0$ for $x > 0$, and $f' < 0$ for $x < 0$, so f strictly decreases on $(-\infty, 0)$ and increases on $(0, +\infty)$.

Actually, even without using Theorem 2, the last argument yields the answer.

(b′) Let

$$f(x) = \ln x \text{ on } (0, +\infty).$$

Then

$$f'(x) = \frac{1}{x} > 0 \text{ on all of } (0, +\infty).$$

This shows that f strictly increases everywhere and hence can have no maximum or minimum anywhere. The same follows by the second part of Theorem 2, with $n = 1$.

(b″) In Example (b′), consider also

$$f''(x) = -\frac{1}{x^2} < 0.$$

In this case, f'' has no bearing on the existence of a maximum or minimum because $f' \neq 0$. Still, the formula $f'' < 0$ does have a certain meaning. In fact, *if* $f''(p) < 0$ *and* $f \in \mathrm{CD}^2$ *on* $G_p(\delta)$, then (using the same argument as in Theorem 2) the reader will easily find that

$$f(x) \leq f(p) + f'(p)(x-p) \quad \text{for } x \text{ in some } G_p(\varepsilon),\ 0 < \varepsilon \leq \delta. \tag{10}$$

Since $y = f(p) + f'(p)(x-p)$ is the equation of the *tangent* at p, it follows that $f(x) \leq y$; i.e., *near* p the curve lies *below* the tangent at p.

Similarly, $f''(p) > 0$ and $f \in \mathrm{CD}^2$ on $G_p(\delta)$ implies that the curve near p lies *above* the tangent.

Problems on Taylor's Theorem

1. Complete the proofs of Theorems 1, 1′, and 2.

2. Verify Note 1 and Examples (b) and (b″).

3. Taking $g(t) = (a-t)^s$, $s > 0$, in (6), find

$$R_n = \frac{f^{(n+1)}(q)}{n!\, s}(x-p)^s(x-q)^{n+1-s} \quad (\textit{Schloemilch–Roche} \text{ remainder}).$$

Obtain (5′) and (5″) from it.

4. Prove that P_n (as defined) is the only polynomial of degree n such that

$$f^{(k)}(p) = P_n^{(k)}(p), \quad k = 0, 1, \ldots, n.$$

[Hint: Differentiate P_n n times to verify that it satisfies this property.

For uniqueness, suppose this also holds for

$$P(x) = \sum_{k-0}^{n} a_k (x-p)^k.$$

Differentiate P n times to show that

$$P^{(k)}(p) = f^{(k)}(p) = a_k k!,$$

so $P = P_n$. (Why?)]

5. With P_n as defined, prove that if f is n times differentiable at p, then

$$f(x) - P_n(x) = o((x-p)^n) \text{ as } x \to p$$

(Taylor's theorem with *Peano remainder* term).

[Hint: Let $R(x) = f(x) - P_n(x)$ and

$$\delta(x) = \frac{R(x)}{(x-p)^n} \text{ with } \delta(p) = 0.$$

Using the "simplified" L'Hôpital rule (Problem 3 in §3) repeatedly n times, prove that $\lim_{x\to p} \delta(x) = 0$.]

6. Use Theorem 1′ with $p = 0$ to verify the following expansions, and prove that $\lim_{n\to\infty} R_n = 0$.

(a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots - \frac{(-1)^m x^{2m-1}}{(2m-1)!} + \frac{(-1)^m x^{2m+1}}{(2m+1)!} \cos \theta_m x$ for all $x \in E^1$;

(b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^m x^{2m}}{(2m)!} - \frac{(-1)^m x^{2m+2}}{(2m+2)!} \sin \theta_m x$ for all $x \in E^1$.

[Hints: Let $f(x) = \sin x$ and $g(x) = \cos x$. Induction shows that

$$f^{(n)}(x) = \sin\Big(x + \frac{n\pi}{2}\Big) \text{ and } g^{(n)}(x) = \cos\Big(x + \frac{n\pi}{2}\Big).$$

Using formula (5′), prove that

$$|R_n(x)| \le \left|\frac{x^{n+1}}{(n+1)!}\right| \to 0.$$

Indeed, $x^n/n!$ is the general term of a *convergent* series

$$\sum \frac{x^n}{n!} \quad \text{(see Chapter 4, §13, Example (d)).}$$

Thus $x^n/n! \to 0$ by Theorem 4 of the same section.]

7. For *any* $s \in E^1$ and $n \in N$, define

$$\binom{s}{n} = \frac{s(s-1)\cdots(s-n+1)}{n!} \text{ with } \binom{s}{0} = 1.$$

Then prove the following.

(i) $\lim\limits_{n\to\infty} n\binom{s}{n} = 0$ if $s > 0$.

(ii) $\lim\limits_{n\to\infty} \binom{s}{n} = 0$ if $s > -1$.

(iii) For *any* fixed $s \in E^1$ and $x \in (-1, 1)$,

$$\lim_{n\to\infty} \binom{s}{n} n x^n = 0;$$

hence

$$\lim_{n\to\infty} \binom{s}{n} x^n = 0.$$

[Hints: (i) Let $a_n = \left|n\binom{s}{n}\right|$. Verify that

$$a_n = |s| \left|1 - \frac{s}{1}\right| \left|1 - \frac{s}{2}\right| \cdots \left|1 - \frac{s}{n-1}\right|.$$

If $s > 0$, $\{a_n\}\downarrow$ for $n > s+1$, so we may put $L = \lim a_n = \lim a_{2n} \ge 0$. (Explain!) Prove that

$$\frac{a_{2n}}{a_n} < \left|1 - \frac{s}{2n}\right|^n \to e^{-\frac{1}{2}s} \quad \text{as } n \to \infty,$$

so for large n,

$$\frac{a_{2n}}{a_n} < e^{-\frac{1}{2}s} + \varepsilon; \text{ i.e., } a_{2n} < (e^{-\frac{1}{2}s} + \varepsilon)a_n.$$

With ε fixed, let $n \to \infty$ to get $L \le (e^{-\frac{1}{2}s} + \varepsilon)L$. Then with $\varepsilon \to 0$, obtain $Le^{\frac{1}{2}s} \le L$. As $e^{\frac{1}{2}s} > 1$ (for $s > 0$), this implies $L = 0$, as claimed.

(ii) For $s > -1$, $s + 1 > 0$, so by (i),

$$(n+1)\binom{s+1}{n+1} \to 0; \text{ i.e., } (s+1)\binom{s}{n} \to 0. \quad \text{(Why?)}$$

(iii) Use the ratio test to show that the series $\sum \binom{s}{n} nx^n$ converges when $|x| < 1$. Then apply Theorem 4 of Chapter 4, §13.]

8. Continuing Problems 6 and 7, prove that

$$(1+x)^s = \sum_{k=0}^{n} \binom{s}{k} x^k + R_n(x),$$

where $R_n(x) \to 0$ if either $|x| < 1$, or $x = 1$ and $s > -1$, or $x = -1$ and $s > 0$.

[Hints: (a) If $0 \le x \le 1$, use $(5')$ for

$$R_{n-1}(x) = \binom{s}{n} x^n (1 + \theta_n x)^{s-n}, \quad 0 < \theta_n < 1. \text{ (Verify!)}$$

Deduce that $|R_{n-1}(x)| \le \left|\binom{s}{n} x^n\right| \to 0$. Use Problem 7(iii) if $|x| < 1$ or Problem 7(ii) if $x = 1$.

(b) If $-1 \le x < 0$, write $(5'')$ as

$$R_{n-1}(x) = \binom{s}{n} nx^n (1 + \theta'_n x)s^{-1} \left(\frac{1 - \theta'_n}{1 + \theta'_n x}\right)^{n-1}. \quad \text{(Check!)}$$

As $-1 \le x < 0$, the *last* fraction is ≤ 1. (Why?) Also,

$$(1 + \theta'_n x)^{s-1} \le 1 \text{ if } s > 1, \text{ and } \le (1+x)^{s-1} \text{ if } s \le 1.$$

Thus, with x fixed, these expressions are *bounded*, while $\binom{s}{n} nx^n \to 0$ by Problem 7(i) or (iii). Deduce that $R_{n-1} \to 0$, hence $R_n \to 0$.]

9. Prove that

$$\ln(1+x) = \sum_{k=1}^{n} (-1)^{k+1} \frac{x^k}{k} + R_n(x),$$

where $\lim_{n\to\infty} R_n(x) = 0$ if $-1 < x \le 1$.

[Hints: If $0 \le x \le 1$, use formula $(5')$.

If $-1 < x < 0$, use formula (6) with $g(t) = \ln(1+t)$ to obtain

$$R_n(x) = \frac{\ln(1+x)}{(-1)^n} \left(\frac{1 - \theta_n}{1 + \theta_n x} \cdot x\right)^n.$$

Proceed as in Problem 8.]

10. Prove that if $f\colon E^1 \to E^*$ is of class CD^1 on $[a, b]$ and if $-\infty < f'' < 0$ on (a, b), then for each $x_0 \in (a, b)$,

$$f(x_0) > \frac{f(b) - f(a)}{b - a}(x_0 - a) + f(a);$$

i.e., *the curve* $y = f(x)$ *lies above the secant* through $(a, f(a))$ and $(b, f(b))$.

[Hint: This formula is equivalent to

$$\frac{f(x_0) - f(a)}{x_0 - a} > \frac{f(b) - f(a)}{b - a},$$

i.e., the average of f' on $[a, x_0]$ is strictly greater than the average of f' on $[a, b]$, which follows because f' *decreases* on (a, b). (Explain!)]

11. Prove that if a, b, r, and s are positive reals and $r + s = 1$, then

$$a^r b^s \leq ra + sb.$$

(This inequality is important for the theory of so-called L^p*-spaces*.)

[Hints: If $a = b$, all is trivial.

Therefore, assume $a < b$. Then

$$a = (r + s)a < ra + sb < b.$$

Use Problem 10 with $x_0 = ra + sb \in (a, b)$ and

$$f(x) = \ln x,\ f''(x) = -\frac{1}{x^2} < 0.$$

Verify that

$$x_0 - a = x_0 - (r + s)a = s(b - a)$$

and $r \cdot \ln a = (1 - s) \ln a$; hence deduce that

$$r \cdot \ln a + s \cdot \ln b - \ln a = s(\ln b - \ln a) = s(f(b) - f(a)).$$

After substitutions, obtain

$$f(x_0) = \ln(ra + sb) > r \cdot \ln a + s \cdot \ln b = \ln(a^r b^s).]$$

12. Use Taylor's theorem (Theorem 1$'$) to prove the following inequalities:

(a) $\sqrt[3]{1 + x} < 1 + \dfrac{x}{3}$ if $x > -1$, $x \neq 0$.

(b) $\cos x > 1 - \dfrac{1}{2}x^2$ if $x \neq 0$.

(c) $\dfrac{x}{1 + x^2} < \arctan x < x$ if $x > 0$.

(d) $x > \sin x > x - \dfrac{1}{6}x^3$ if $x > 0$.

§7. The Total Variation (Length) of a Function $f\colon E^1 \to E$

The question that we shall consider now is how to define reasonably (and precisely) the notion of the *length* of a curve (Chapter 4, §10) described by a function $f\colon E^1 \to E$ over an interval $I = [a, b]$, i.e., $f[I]$.

We proceed as follows (see Figure 24).[1]

Subdivide $[a, b]$ by a finite set of points $P = \{t_0, t_1, \ldots, t_m\}$, with

$$a = t_0 \le t_1 \le \cdots \le t_m = b;$$

P is called a *partition* of $[a, b]$. Let

$$q_i = f(t_i), \quad i = 1, 2, \ldots, m,$$

and, for $i = 1, 2, \ldots, m$,

$$\begin{aligned}\Delta_i f &= q_i - q_{i-1}\\ &= f(t_i) - f(t_{i-1}).\end{aligned}$$

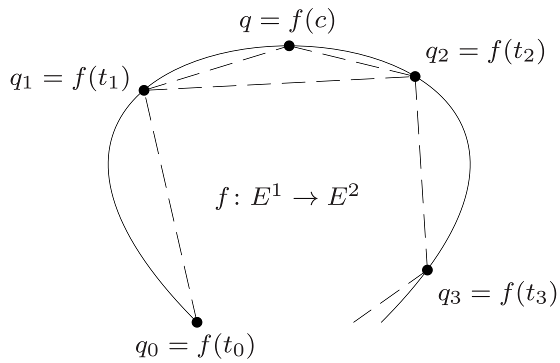

FIGURE 24

We also define

$$S(f, P) = \sum_{i=1}^{m} |\Delta_i f| = \sum_{i=1}^{m} |q_i - q_{i-1}|.$$

Geometrically, $|\Delta_i f| = |q_i - q_{i-1}|$ is the length of the line segment $L[q_{i-1}, q_i]$ in E, and $S(f, P)$ is the sum of such lengths, i.e., the length of the *polygon*

$$W = \bigcup_{i=1}^{m} L[q_{i-1}, q_i]$$

inscribed into $f[I]$; we denote it by

$$\ell W = S(f, P).$$

Now suppose we add a new partition point c, with

$$t_{i-1} \le c \le t_i.$$

Then we obtain a new partition

$$P_c = \{t_0, \ldots, t_{i-1}, c, t_i, \ldots, t_m\},$$

called a *refinement* of P, and a new inscribed polygon W_c in which $L[q_{i-1}, q_i]$ is replaced by *two* segments, $L[q_{i-1}, q]$ and $L[q, q_i]$, where $q = f(c)$; see Figure 24. Accordingly, the term $|\Delta_i f| = |q_i - q_{i-1}|$ in $S(f, P)$ is replaced by

$$|q_i - q| + |q - q_{i-1}| \ge |q_i - q_{i-1}| \quad \text{(triangle law)}.$$

[1] Note that this method works even if f is *discontinuous*.

It follows that

$$S(f, P) \le S(f, P_c); \text{ i.e., } \ell W \le \ell W_c.$$

Hence we obtain the following result.

Corollary 1. *The sum $S(f, P) = \ell W$ cannot decrease when P is refined.*

Thus when new partition points are added, $S(f, P)$ *grows* in general; i.e., *it approaches some supremum value* (finite or not). Roughly speaking, the inscribed polygon W gets "closer" to the curve. It is natural to define the desired length of the curve to be the *lub of all lengths* ℓW, i.e., of all sums $S(f, P)$ resulting from the various partitions P. This supremum is also called the *total variation* of f over $[a, b]$, denoted $V_f[a, b]$.[2]

Definition 1.

Given any function $f\colon E^1 \to E$, and $I = [a, b] \subset E^1$, we set

$$V_f[I] = V_f[a, b] = \sup_P S(f, P) = \sup_P \sum_{i=1}^{m} |f(t_i) - f(t_{i-1})| \ge 0 \text{ in } E^*, \tag{1}$$

where the supremum is over all partitions $P = \{t_0, \ldots, t_m\}$ of I. We call $V_f[I]$ the *total variation*, or *length*, of f on I. Briefly, we denote it by V_f.

Note 1. If f is continuous on $[a, b]$, the image set $A = f[I]$ is an *arc* (Chapter 4, §10). It is customary to call $V_f[I]$ the *length of that arc*, denoted $\ell_f A$ or briefly ℓA. Note, however, that there may well be another function g, continuous on an interval J, such that $g[J] = A$ but $V_f[I] \ne V_g[J]$, and so $\ell_f A \ne \ell_g A$. Thus it is safer to say "the length of A *as described by f on I*." Only for *simple arcs* (where f is one to one), is "ℓA" unambiguous. (See Problems 6–8.)

In the propositions below, f is an *arbitrary* function, $f\colon E^1 \to E$.

Theorem 1 (additivity of V_f). *If $a \le c \le b$, then*

$$V_f[a, b] = V_f[a, c] + V_f[c, b];$$

i.e., the length of the whole equals the sum of the lengths of the parts.

Proof. Take any partition $P = \{t_0, \ldots, t_m\}$ of $[a, b]$. If $c \notin P$, refine P to

$$P_c = \{t_0, \ldots, t_i, c, t_i, \ldots, t_m\}.$$

Then by Corollary 1, $S(f, P) \le S(f, P_c)$.

Now P_c splits into partitions of $[a, c]$ and $[c, b]$, namely,

$$P' = \{t_0, \ldots, t_{i-1}, c\} \text{ and } P'' = \{c, t_i, \ldots, t_m\},$$

[2] We also call it the *length of f* over $[a, b]$. Observe that, for *real* $f\colon E^1 \to E^1$, this is not the length of the *graph* in the usual sense (which is a curve *in* E^2). See the end of §8.

so that

$$S(f,\, P') + S(f,\, P'') = S(f,\, P_c). \quad \text{(Verify!)}$$

Hence (as V_f is the *lub* of the corresponding sums),

$$V_f[a,\, c] + V_f[c,\, d] \geq S(f,\, P_c) \geq S(f,\, P).$$

As P is an *arbitrary* partition of $[a,\, b]$, we also have

$$V_f[a,\, c] + V_f[c,\, b] \geq \sup S(f,\, P) = V_f[a,\, b].$$

Thus it remains to show that, conversely,

$$V_f[a,\, b] \geq V_f[a,\, c] + V_f[c,\, b].$$

The latter is trivial if $V_f[a,\, b] = +\infty$. Thus assume $V_f[a,\, b] = K < +\infty$. Let P' and P'' be any partitions of $[a,\, c]$ and $[c,\, b]$, respectively. Then $P^* = P' \cup P''$ is a partition of $[a,\, b]$, and

$$S(f,\, P') + S(f,\, P'') = S(f,\, P^*) \leq V_f[a,\, b] = K,$$

whence

$$S(f,\, P') \leq K - S(f,\, P'').$$

Keeping P'' fixed and varying P', we see that $K - S(f,\, P'')$ is an upper bound of *all* $S(f,\, P')$ over $[a,\, c]$. Hence

$$V_f[a,\, c] \leq K - S(f,\, P'')$$

or

$$S(f,\, P'') \leq K - V_f[a,\, c].$$

Similarly, varying P'', we now obtain

$$V_f[c,\, b] \leq K - V_f[a,\, c]$$

or

$$V_f[a,\, c] + V_f[c,\, b] \leq K = V_f[a,\, b],$$

as required. Thus all is proved. □

Corollary 2 (monotonicity of V_f)**.** *If $a \leq c \leq d \leq b$, then*

$$V_f[c,\, d] \leq V_f[a,\, b].$$

Proof. By Theorem 1,

$$V_f[a,\, b] = V_f[a,\, c] + V_f[c,\, d] + V_f[d,\, b] \geq V_f[c,\, d]. \quad \square$$

Definition 2.

If $V_f[a, b] < +\infty$, we say that f is *of bounded variation* on $I = [a, b]$, and that the set $f[I]$ is *rectifiable* (*by f on I*).

Corollary 3. *For each $t \in [a, b]$,*

$$|f(t) - f(a)| \le V_f[a, b].$$

Hence if f is of bounded variation on $[a, b]$, it is bounded on $[a, b]$.

Proof. If $t \in [a, b]$, let $P = \{a, t, b\}$, so

$$|f(t) - f(a)| \le |f(t) - f(a)| + |f(b) - f(t)| = S(f, P) \le V_f[a, b],$$

proving our first assertion.[3] Hence

$$(\forall\, t \in [a, b]) \quad |f(t)| \le |f(t) - f(a)| + |f(a)| \le V_f[a, b] + |f(a)|.$$

This proves the second assertion. □

Note 2. Neither boundedness, nor continuity, nor differentiability of f on $[a, b]$ implies $V_f[a, b] < +\infty$, but *boundedness of f' does*. (See Problems 1 and 3.)

Corollary 4. *A function f is finite and constant on $[a, b]$ iff $V_f[a, b] = 0$.*

The proof is left to the reader. (Use Corollary 3 and the definitions.)

Theorem 2. *Let f, g, h be real or complex (or let f and g be vector valued and h scalar valued). Then on any interval $I = [a, b]$, we have*

(i) $V_{|f|} \le V_f$;

(ii) $V_{f \pm g} \le V_f + V_g$; *and*

(iii) $V_{hf} \le sV_f + rV_h$, *with* $r = \sup_{t \in I} |f(t)|$ *and* $s = \sup_{t \in I} |h(t)|$.

Hence if f, g, and h are of bounded variation on I, so are $f \pm g$, hf, and $|f|$.

Proof. We first prove (iii).

Take any partition $P = \{t_0, \dots, t_m\}$ of I. Then

$$\begin{aligned} |\Delta_i hf| &= |h(t_i)f(t_i) - h(t_{i-1})f(t_{i-1})| \\ &\le |h(t_i)f(t_i) - h(t_{i-1})f(t_i)| + |h(t_{i-1})f(t_i) - h(t_{i-1})f(t_{i-1})| \\ &= |f(t_i)||\Delta_i h| + |h(t_{i-1})||\Delta_i f| \\ &\le r|\Delta_i h| + s|\Delta_i f|. \end{aligned}$$

Adding these inequalities, we obtain

$$S(hf, P) \le r \cdot S(h, P) + s \cdot S(f, P) \le rV_h + sV_f.$$

[3] By our conventions, it also follows that $|f(a)|$ is a *finite* constant, and so is $V_f[a, b] + |f(a)|$ if $V_f[a, b] < +\infty$.

As this holds for *all* sums $S(hf, P)$, it holds for their supremum, so

$$V_{hf} = \sup S(hf, P) \le rV_h + sV_f,$$

as claimed.

Similarly, (i) follows from

$$\big||f(t_i)| - |f(t_{i-1})|\big| \le |f(t_i) - f(t_{i-1})|.$$

The analogous proof of (ii) is left to the reader.

Finally, (i)–(iii) imply that V_f, $V_{f\pm g}$, and V_{hf} are *finite* if V_f, V_g, and V_h are. This proves our *last* assertion. □

Note 3. Also f/h is of bounded variation on I if f and h are, *provided* h *is bounded away from* 0 *on* I; i.e.,

$$(\exists\, \varepsilon > 0) \quad |h| \ge \varepsilon \text{ on } I.$$

(See Problem 5.)

Special theorems apply in case the range space E is E^1 or E^n (*or C^n).

Theorem 3.

(i) *A* real *function* f *is of bounded variation on* $I = [a, b]$ *iff* $f = g - h$ *for some nondecreasing real functions* g *and* h *on* I.

(ii) *If* f *is real and monotone on* I, *it is of bounded variation there.*

Proof. We prove (ii) first.

Let $f\uparrow$ on I. If $P = \{t_0, \ldots, t_m\}$, then

$$t_i \ge t_{i-1} \text{ implies } f(t_i) \ge f(t_{i-1}).$$

Hence $\Delta_i f \ge 0$; i.e., $|\Delta_i f| = \Delta_i f$. Thus

$$\begin{aligned} S(f, P) &= \sum_{i=1}^{m} |\Delta_i f| = \sum_{i=1}^{m} \Delta_i f = \sum_{i=1}^{m} [f(t_i) - f(t_{i-1})] \\ &= f(t_m) - f(t_0) = f(b) - f(a) \end{aligned}$$

for *any* P. (Verify!) This implies that also

$$V_f[I] = \sup S(f, P) = f(b) - f(a) < +\infty.$$

Thus (ii) is proved.

Now for (i), let $f = g - h$ with $g\uparrow$ and $h\uparrow$ on I. By (ii), g and h are of bounded variation on I. Hence so is $f = g - h$ by Theorem 2 (last clause).

Conversely, suppose $V_f[I] < +\infty$. Then define

$$g(x) = V_f[a, x],\ x \in I, \text{ and } h = g - f \text{ on } I,$$

so $f = g - h$, and it only remains to show that $g\uparrow$ and $h\uparrow$.

To prove it, let $a \le x \le y \le b$. Then Theorem 1 yields

$$V_f[a,\, y] - V_f[a,\, x] = V_f[x,\, y];$$

i.e.,

$$g(y) - g(x) = V_f[x,\, y] \ge |f(y) - f(x)| \ge 0 \quad \text{(by Corollary 3).} \tag{2}$$

Hence $g(y) \ge g(x)$. Also, as $h = g - f$, we have

$$\begin{aligned} h(y) - h(x) &= g(y) - f(y) - [g(x) - f(x)] \\ &= g(y) - g(x) - [f(y) - f(x)] \\ &\ge 0 \quad \text{by (2).} \end{aligned}$$

Thus $h(y) \ge h(x)$. We see that $a \le x \le y \le b$ implies $g(x) \le g(y)$ and $h(x) \le h(y)$, so $h\uparrow$ and $g\uparrow$, indeed. □

Theorem 4.

(i) *A function $f\colon E^1 \to E^n$ ($^*C^n$) is of bounded variation on $I = [a,\, b]$ iff all of its components ($f_1,\, f_2,\, \ldots,\, f_n$) are.*

(ii) *If this is the case, then finite limits $f(p^+)$ and $f(q^-)$ exist for every $p \in [a,\, b)$ and $q \in (a,\, b]$.*

Proof.

(i) Take any partition $P = \{t_0,\, \ldots,\, t_m\}$ of I. Then

$$|f_k(t_i) - f_k(t_{i-1})|^2 \le \sum_{j=1}^{n} |f_j(t_i) - f_j(t_{i-1})|^2 = |f(t_i) - f(t_{i-1})|^2;$$

i.e., $|\Delta_i f_k| \le |\Delta_i f|$, $i = 1,\, 2,\, \ldots,\, m$. Thus

$$(\forall\, P) \quad S(f_k,\, P) \le S(f,\, P) \le V_f,$$

and $V_{f_k} \le V_f$ follows. Thus

$$V_f < +\infty \text{ implies } V_{f_k} < +\infty, \quad k = 1,\, 2,\, \ldots,\, n.$$

The converse follows by Theorem 2 since $f = \sum_{k=1}^{n} f_k \vec{e}_k$. (Explain!)

(ii) For *real monotone* functions, $f(p^+)$ and $f(q^-)$ exist by Theorem 1 of Chapter 4, §5. This also applies if f is *real and of bounded variation*, for by Theorem 3,

$$f = g - h \text{ with } g\uparrow \text{ and } h\uparrow \text{ on } I,$$

and so

$$f(p^+) = g(p^+) - h(p^+) \text{ and } f(q^-) = g(q^-) - h(q^-) \text{ exist.}$$

The limits are finite since f is bounded on I by Corollary 3.

Via *components* (Theorem 2 of Chapter 4, §3), this also applies to functions $f\colon E^1 \to E^n$. (Why?) In particular, (ii) applies to *complex* functions (treat C as E^2) (*and so it extends to functions $f\colon E^1 \to C^n$ as well). □

We also have proved the following corollary.

Corollary 5. *A complex function $f\colon E^1 \to C$ is of bounded variation on $[a, b]$ iff its real and imaginary parts are.* (*See Chapter* 4, §3, *Note* 5.)

Problems on Total Variation and Graph Length

1. In the following cases show that $V_f[I] = +\infty$, though f is bounded on I. (In case (iii), f is continuous, and in case (iv), it is even differentiable on I.)

(i) For $I = [a, b]$ $(a < b)$, $f(x) = \begin{cases} 1 & \text{if } x \in R \text{ (rational), and} \\ 0 & \text{if } x \in E^1 - R. \end{cases}$

(ii) $f(x) = \sin \dfrac{1}{x}$; $f(0) = 0$; $I = [a, b]$, $a \le 0 \le b$, $a < b$.

(iii) $f(x) = x \cdot \sin \dfrac{\pi}{2x}$; $f(0) = 0$; $I = [0, 1]$.

(iv) $f(x) = x^2 \sin \dfrac{1}{x^2}$; $f(0) = 0$; $I = [0, 1]$.

[Hints: (i) For any m there is P, with

$$|\Delta_i f| = 1, \quad i = 1, 2, \ldots, m,$$

so $S(f, P) = m \to +\infty$.

(iii) Let

$$P_m = \left\{0, \frac{1}{m}, \frac{1}{m-1}, \ldots, \frac{1}{2}, 1\right\}.$$

Prove that $S(f, P_m) \ge \sum_{k=1}^{m} \frac{1}{k} \to +\infty$.]

2. Let $f\colon E^1 \to E^1$ be monotone on each of the intervals

$$[a_{k-1}, a_k], \quad k = 1, \ldots, n \quad (\text{“}\textit{piecewise monotone}\text{”}).$$

Prove that

$$V_f[a_0, a_n] = \sum_{k=1}^{n} |f(a_k) - f(a_{k-1})|.$$

In particular, show that this applies if $f(x) = \sum_{i=1}^{n} c_i x^i$ (polynomial), with $c_i \in E^1$.

[Hint: It is known that a polynomial of degree n has at most n real roots. Thus it is piecewise monotone, for its derivative vanishes at *finitely* many points (being of degree $n - 1$). Use Theorem 1 in §2.]

⇒3. Prove that if f is finite and relatively continuous on $I = [a, b]$, with a *bounded* derivative, $|f'| \le M$, on $I - Q$ (see §4), then

$$V_f[a, b] \le M(b - a).$$

However, we may have $V_f[I] < +\infty$, and yet $|f'| = +\infty$ at some $p \in I$.
[Hint: Take $f(x) = \sqrt[3]{x}$ on $[-1, 1]$.]

4. Complete the proofs of Corollary 4 and Theorems 2 and 4.

5. Prove Note 3.
[Hint: If $|h| \ge \varepsilon$ on I, show that

$$\left|\frac{1}{h(t_i)} - \frac{1}{h(t_{i-1})}\right| \le \frac{|\Delta_i h|}{\varepsilon^2}$$

and hence

$$S\Big(\frac{1}{h}, P\Big) \le \frac{S(h, P)}{\varepsilon^2} \le \frac{V_h}{\varepsilon^2}.$$

Deduce that $\frac{1}{h}$ is of bounded variation on I if h is. Then apply Theorem 2(iii) to $\frac{1}{h} \cdot f$.]

6. Let $g\colon E^1 \to E^1$ (real) and $f\colon E^1 \to E$ be relatively continuous on $J = [c, d]$ and $I = [a, b]$, respectively, with $a = g(c)$ and $b = g(d)$. Let

$$h = f \circ g.$$

Prove that if g is one to one on J, then

(i) $g[J] = I$, so f and h describe one and the same arc $A = f[I] = h[J]$;

(ii) $V_f[I] = V_h[J]$; i.e., $\ell_f A = \ell_h A$.

[Hint for (ii): Given $P = \{a = t_0, \ldots, t_m = b\}$, show that the points $s_i = g^{-1}(t_i)$ form a partition P' of $J = [c, d]$, with $S(h, P') = S(f, P)$. Hence deduce $V_f[I] \le V_h[J]$.

Then prove that $V_h[J] \le V_f[I]$, taking an *arbitrary* $P' = \{c = s_0, \ldots, s_m = d\}$, and *defining* $P = \{t_0, \ldots, t_m\}$, with $t_i = g(s_i)$. What if $g(c) = b$, $g(d) = a$?]

7. Prove that if $f, h\colon E^1 \to E$ are relatively continuous and *one to one* on $I = [a, b]$ and $J = [c, d]$, respectively, and if

$$f[I] = h[J] = A$$

(i.e., f and h describe the same *simple* arc A), then

$$\ell_f A = \ell_h A.$$

Thus for *simple* arcs, $\ell_f A$ is independent of f.
[Hint: Define $g\colon J \to E^1$ by $g = f^{-1} \circ h$. Use Problem 6 and Chapter 4, §9, Theorem 3. First check that Problem 6 works also if $g(c) = b$ and $g(d) = a$, i.e., $g\downarrow$ on J.]

8. Let $I = [0, 2\pi]$ and define $f, g, h\colon E^1 \to E^2$ (C) by

$$f(x) = (\sin x, \cos x),$$
$$g(x) = (\sin 3x, \cos 3x),$$
$$h(x) = \left(\sin \frac{1}{x}, \cos \frac{1}{x}\right) \text{ with } h(0) = (0, 1).$$

Show that $f[I] = g[I] = h[I]$ (the *unit circle*; call it A), yet $\ell_f A = 2\pi$, $\ell_g A = 6\pi$, while $V_h[I] = +\infty$. (Thus the result of Problem 7 fails for closed curves and *nonsimple* arcs.)

9. In Theorem 3, define two functions $G, H\colon E^1 \to E^1$ by

$$G(x) = \frac{1}{2}[V_f[a, x] + f(x) - f(a)]$$

and

$$H(x) = G(x) - f(x) + f(a).$$

(G and H are called, respectively, the *positive* and *negative variation functions* for f.) Prove that

(i) $G\uparrow$ and $H\uparrow$ on $[a, b]$;

(ii) $f(x) = G(x) - [H(x) - f(a)]$ (thus the functions f and g of Theorem 3 are *not unique*);

(iii) $V_f[a, x] = G(x) + H(x)$;

(iv) if $f = g - h$, with $g\uparrow$ and $h\uparrow$ on $[a, b]$, then

$$V_G[a, b] \le V_g[a, b], \text{ and } V_H[a, b] \le V_h[a, b];$$

(v) $G(a) = H(a) = 0$.

***10.** Prove that if $f\colon E^1 \to E^n$ (C^n) is of bounded variation on $I = [a, b]$, then f has at most *countably many* discontinuities in I.
[Hint: Apply Problem 5 of Chapter 4, §5. Proceed as in the proof of Theorem 4 in §7. Finally, use Theorem 2 of Chapter 1, §9.]

§8. Rectifiable Arcs. Absolute Continuity

If a function $f\colon E^1 \to E$ is of *bounded variation* (§7) on an interval $I = [a, b]$, we can define a *real* function v_f on I by

$$v_f(x) = V_f[a, x] \text{ (= total variation of } f \text{ on } [a, x]) \text{ for } x \in I;$$

v_f is called the *total variation function*, or *length function*, generated by f on I. Note that $v_f\uparrow$ on I. (Why?) We now consider the case where f is also *relatively continuous* on I, so that the set $A = f[I]$ is a *rectifiable arc* (see §7, Note 1 and Definition 2).

Definition 1.

A function $f\colon E^1 \to E$ is (weakly) *absolutely continuous*[1] on $I = [a, b]$ iff $V_f[I] < +\infty$ and f is relatively continuous on I.

Theorem 1. *The following are equivalent:*

(i) f *is (weakly) absolutely continuous on* $I = [a, b]$;

(ii) v_f *is finite and relatively continuous on* I; *and*

(iii) $(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, x, y \in I \mid 0 \le y - x < \delta)\ V_f[x, y] < \varepsilon.$

Proof. We shall show that (ii) ⇒ (iii) ⇒ (i) ⇒ (ii).

(ii) ⇒ (iii). As $I = [a, b]$ is *compact*, (ii) implies that v_f is *uniformly* continuous on I (Theorem 4 of Chapter 4, §8). Thus

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, x, y \in I \mid 0 \le y - x < \delta) \quad v_f(y) - v_f(x) < \varepsilon.$$

However,

$$v_f(y) - v_f(x) = V_f[a, y] - V_f[a, x] = V_f[x, y]$$

by additivity (Theorem 1 in §7). Thus (iii) follows.

(iii) ⇒ (i). By Corollary 3 of §7, $|f(x) - f(y)| \le V_f[x, y]$. Therefore, (iii) implies that

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, x,\, y \in I \mid |x - y| < \delta) \quad |f(x) - f(y)| < \varepsilon,$$

and so f is relatively (even uniformly) continuous on I.

Now with ε and δ as in (iii), take a partition $P = \{t_0, \ldots, t_m\}$ of I so fine that

$$t_i - t_{i-1} < \delta, \quad i = 1, 2, \ldots, m.$$

Then $(\forall\, i)\ V_f[t_{i-1}, t_i] < \varepsilon$. Adding up these m inequalities and using the additivity of V_f, we obtain

$$V_f[I] = \sum_{i=1}^{m} V_f[t_{i-1}, t_i] < m\varepsilon < +\infty.$$

Thus (i) follows, by definition.

That (i) ⇒ (ii) is given as the next theorem. □

[1] In this section, we use this notion in a *weaker* sense than customary. The usual stronger version is given in Problem 2. We study it in Volume 2, Chapter 7, §11.

Theorem 2. *If $V_f[I] < +\infty$ and if f is relatively continuous at some $p \in I$ (over $I = [a, b]$), then the same applies to the length function v_f.*

Proof. We consider *left* continuity first, with $a < p \le b$.

Let $\varepsilon > 0$. By assumption, there is $\delta > 0$ such that

$$|f(x) - f(p)| < \frac{\varepsilon}{2} \text{ when } |x - p| < \delta \text{ and } x \in [a, p].$$

Fix any such x. Also, $V_f[a, p] = \sup_P S(f, P)$ over $[a, p]$. Thus

$$V_f[a, p] - \frac{\varepsilon}{2} < \sum_{i=1}^{k} |\Delta_i f|$$

for some partition

$$P = \{t_0 = a, \ldots, t_{k-1}, t_k = p\} \text{ of } [a, p]. \text{ (Why?)}$$

We may assume $t_{k-1} = x$, x as above. (If $t_{k-1} \ne x$, *add* x to P.) Then

$$|\Delta_k f| = |f(p) - f(x)| < \frac{\varepsilon}{2},$$

and hence

$$V_f[a, p] - \frac{\varepsilon}{2} < \sum_{i=1}^{k-1} |\Delta_i f| + |\Delta_k f| < \sum_{i=1}^{k-1} |\Delta_i f| + \frac{\varepsilon}{2} \le V_f[a, t_{k-1}] + \frac{\varepsilon}{2}. \tag{1}$$

However,

$$V_f[a, p] = v_f(p)$$

and

$$V_f[a, t_{k-1}] = V_f[a, x] = v_f(x).$$

Thus (1) yields

$$|v_f(p) - v_f(x)| = V_f[a, p] - V_f[a, x] < \varepsilon \text{ for } x \in [a, p] \text{ with } |x - p| < \delta.$$

This shows that v_f is left continuous at p.

Right continuity is proved similarly on noting that

$$v_f(x) - v_f(p) = V_f[p, b] - V_f[x, b] \text{ for } p \le x < b. \text{ (Why?)}$$

Thus v_f is, indeed, relatively continuous at p. Observe that v_f is also of bounded variation on I, being monotone and finite (see Theorem 3(ii) of §7).

This completes the proof of both Theorem 2 and Theorem 1. □

We also have the following.

Corollary 1. *If f is real and absolutely continuous on $I = [a, b]$ (weakly), so are the nondecreasing functions g and h ($f = g - h$) defined in Theorem 3 of §7.*

Indeed, the function g as defined there is simply v_f. Thus it is relatively continuous and finite on I by Theorem 1. Hence so also is $h = f - g$. Both are of bounded variation (monotone!) and hence absolutely continuous (weakly).

Note 1. The proof of Theorem 1 shows that (*weak*) *absolute continuity implies uniform continuity*. The converse fails, however (see Problem 1(iv) in §7).

We now apply our theory to *antiderivatives* (integrals).

Corollary 2. *If $F = \int f$ on $I = [a, b]$ and if f is bounded ($|f| \le K \in E^1$) on $I - Q$ (Q countable), then F is weakly absolutely continuous on I.*

(Actually, even the stronger variety of absolute continuity follows. See Chapter 7, §11, Problem 17).

Proof. By definition, $F = \int f$ is finite and relatively continuous on I, so we only have to show that $V_F[I] < +\infty$. This, however, easily follows by Problem 3 of §7 on noting that $F' = f$ on $I - S$ (S countable). Details are left to the reader. $\square$

Our next theorem expresses arc length in the form of an integral.

Theorem 3. *If $f\colon E^1 \to E$ is continuously differentiable on $I = [a, b]$ (§6), then $v_f = \int |f'|$ on I and*

$$V_f[a, b] = \int_a^b |f'|.$$

Proof. Let $a < p < x \le b$, $\Delta x = x - p$, and

$$\Delta v_f = v_f(x) - v_f(p) = V_f[p, x]. \quad \text{(Why?)}$$

As a first step, we shall show that

$$\frac{\Delta v_f}{\Delta x} \le \sup_{[p,x]} |f'|. \tag{2}$$

For any partition $P = \{p = t_0, \ldots, t_m = x\}$ of $[p, x]$, we have

$$S(f, P) = \sum_{i=1}^{m} |\Delta_i f| \le \sum_{i=1}^{m} \sup_{[t_{i-1}, t_i]} |f'| \, (t_i - t_{i-1}) \le \sup_{[p,x]} |f'| \, \Delta x.$$

Since this holds for *any* partition P, we have

$$V_f[p, x] \le \sup_{[p,x]} |f'| \, \Delta x,$$

which implies (2).

On the other hand,

$$\Delta v_f = V_f[p,\, x] \geq |f(x) - f(p)| = |\Delta f|.$$

Combining, we get

$$\left|\frac{\Delta f}{\Delta x}\right| \leq \frac{\Delta v_f}{\Delta x} \leq \sup_{[p,x]} |f'| < +\infty \tag{3}$$

since f' is relatively continuous on $[a, b]$, hence also uniformly continuous and bounded. (Here we assumed $a < p < x \leq b$. However, (3) holds also if $a \leq x < p < b$, with $\Delta v_f = -V[x,\, p]$ and $\Delta x < 0$. Verify!)

Now

$$\big||f'(p)| - |f'(x)|\big| \leq |f'(p) - f'(x)| \to 0 \quad \text{as } x \to p,$$

so, taking limits as $x \to p$, we obtain

$$\lim_{x \to p} \frac{\Delta v_f}{\Delta x} = |f'(p)|.$$

Thus v_f is differentiable at each p in $(a,\, b)$, with $v_f'(p) = |f'(p)|$. Also, v_f is relatively continuous and finite on $[a,\, b]$ (by Theorem 1).[2] Hence $v_f = \int |f'|$ on $[a,\, b]$, and we obtain

$$\int_a^b |f'| = v_f(b) - v_f(a) = V_f[a,\, b], \text{ as asserted.} \quad \square \tag{4}$$

Note 2. If the range space E is E^n (*or C^n), f has n components

$$f_1,\, f_2,\, \ldots,\, f_n.$$

By Theorem 5 in §1, $f' = (f_1',\, f_2',\, \ldots,\, f_n')$, so

$$|f'| = \sqrt{\sum_{k=1}^{n} |f_k'|^2},$$

and we get

$$V_f[a,\, b] = \int_a^b \sqrt{\sum_{k=1}^{n} |f_k'|^2} = \int_a^b \sqrt{\sum_{k=1}^{n} |f_k'(t)|^2}\, dt \quad \text{(classical notation).} \tag{5}$$

In particular, for complex functions, we have (see Chapter 4, §3, Note 5)

$$V_f[a,\, b] = \int_a^b \sqrt{f_{\rm re}'(t)^2 + f_{\rm im}'(t)^2}\, dt. \tag{6}$$

In practice, formula (5) is used when a curve is given *parametrically* by

$$x_k = f_k(t), \quad k = 1,\, 2,\, \ldots,\, n,$$

[2] Note that (3) implies the *finiteness* of $v_f(p)$ and $v_f(x)$.

with the f_k differentiable on $[a, b]$. Curves *in* E^2 are often given in *nonparametric* form as

$$y = F(x), \quad F\colon E^1 \to E^1.$$

Here $F[I]$ is *not* the desired curve but simply a set in E^1. To apply (5) here, we first replace "$y = F(x)$" by suitable parametric equations,

$$x = f_1(t) \text{ and } y = f_2(t);$$

i.e., we introduce a function $f\colon E^1 \to E$, with $f = (f_1, f_2)$. An obvious (but not the only) way of achieving it is to set

$$x = f_1(t) = t \text{ and } y = f_2(t) = F(t)$$

so that $f_1' = 1$ and $f_2' = F'$. Then formula (5) may be written as

$$V_f[a, b] = \int_a^b \sqrt{1 + F'(x)^2}\, dx, \quad f(x) = (x, F(x)). \tag{7}$$

Example.

Find the length of the circle

$$x^2 + y^2 = r^2.$$

Here it is convenient to use the parametric equations

$$x = r\cos t \text{ and } y = r\sin t,$$

i.e., to define $f\colon E^1 \to E^2$ by

$$f(t) = (r\cos t, r\sin t),$$

or, in complex notation,

$$f(t) = re^{ti}.$$

Then the circle is obtained by letting t vary through $[0, 2\pi]$. Thus (5) yields

$$V_f[0, 2\pi] = \int_a^b r\sqrt{\cos^2 t + \sin^2 t}\, dt = r\int_a^b 1\, dt = rt\Big|_0^{2\pi} = 2r\pi.$$

Note that f describes *the same* circle $A = f[I]$ over $I = [0, 4\pi]$. More generally, we could let t vary through any interval $[a, b]$ with $b - a \geq 2\pi$. However, the length, $V_f[a, b]$, would *change* (depending on $b - a$). This is because the circle $A = f[I]$ is not a *simple* arc (see §7, Note 1), so ℓA depends on f and I, and one must be careful in selecting both appropriately.

Problems on Absolute Continuity and Rectifiable Arcs

1. Complete the proofs of Theorems 2 and 3, giving all missing details.

⇒2. Show that f is absolutely continuous (in the weaker sense) on $[a, b]$ *if* for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sum_{i=1}^{m} |f(t_i) - f(s_i)| < \varepsilon \text{ whenever } \sum_{i=1}^{m} (t_i - s_i) < \delta \text{ and}$$
$$a \le s_1 \le t_1 \le s_2 \le t_2 \le \cdots \le s_m \le t_m \le b.$$

(This is absolute continuity in the *stronger* sense.)

3. Prove that v_f is *strictly* monotone on $[a, b]$ iff f is not constant on any nondegenerate subinterval of $[a, b]$.
[Hint: If $x < y$, $V_f[x, y] > 0$, by Corollary 4 of §7].

4. With f, g, h as in Theorem 2 of §7, prove that if f, g, h are absolutely continuous (in the weaker sense) on I, so are $f \pm g$, hf, and $|f|$; so also is f/h if $(\exists\, \varepsilon > 0)\ |h| \ge \varepsilon$ on I.

5. Prove the following:

(i) If f' is finite and $\neq 0$ on $I = [a, b]$, so also is v_f' (with *one-sided* derivatives at the endpoints of the interval); moreover,

$$\left| \frac{f'}{v_f'} \right| = 1 \text{ on } I.$$

Thus show that f'/v_f' is the *tangent unit vector* (see §1).

(ii) Under the same assumptions, $F = f \circ v_f^{-1}$ is differentiable on $J = [0, v_f(b)]$ (with *one-sided* derivatives at the endpoints of the interval) and $F[J] = f[I]$; i.e., F and f describe the same simple arc, with $V_F[I] = V_f[I]$.

Note that this is tantamount to a *change of parameter*. Setting $s = v_f(t)$, i.e., $t = v_f^{-1}(s)$, we have $f(t) = f(v_f^{-1}(s)) = F(s)$, with *the arclength s as parameter*.

§9. Convergence Theorems in Differentiation and Integration

Given

$$F_n = \int f_n \text{ or } F_n' = f_n, \quad n = 1, 2, \ldots,$$

what can one say about $\int \lim f_n$ or $(\lim F_n)'$ if the limits exist? Below we give some answers, for *complete* range spaces E (such as E^n). Roughly, we have

$\lim F_n' = (\lim F_n)'$ on $I - Q$ if

(a) $\{F_n(p)\}$ converges for *at least one* $p \in I$ and

(b) $\{F_n'\}$ converges *uniformly*.

Here I is a finite or infinite interval in E^1 and Q is countable. *We include in Q the endpoints of I* (if any), so $I - Q \subseteq I^0$ (= *interior* of I).

Theorem 1. *Let $F_n\colon E^1 \to E$ $(n = 1, 2, \ldots)$ be finite and relatively continuous on I and differentiable on $I - Q$. Suppose that*

(a) $\lim_{n\to\infty} F_n(p)$ *exists for some* $p \in I$;

(b) $F_n' \to f \neq \pm\infty$ (uniformly) *on $J - Q$ for each* finite *subinterval* $J \subseteq I$;

(c) *E is complete.*

Then

(i) $\lim_{n\to\infty} F_n = F$ *exists uniformly on each finite subinterval* $J \subseteq I$;

(ii) $F = \int f$ *on I; and*

(iii) $(\lim F_n)' = F' = f = \lim_{n\to\infty} F_n'$ *on* $I - Q$.

Proof. Fix $\varepsilon > 0$ and any subinterval $J \subseteq I$ of length $\delta < \infty$, with $p \in J$ (p as in (a)). By (b), $F_n' \to f$ (uniformly) on $J - Q$, so there is a k such that for $m, n > k$,

$$|F_n'(t) - f(t)| < \frac{\varepsilon}{2}, \quad t \in J - Q; \tag{1}$$

hence

$$\sup_{t\in J-Q} |F_m'(t) - F_n'(t)| \leq \varepsilon. \quad \text{(Why?)} \tag{2}$$

Now apply Corollary 1 in §4 to the function $h = F_m - F_n$ on J. Then for each $x \in J$, $|h(x) - h(p)| \leq M|x - p|$, where

$$M \leq \sup_{t\in J-Q} |h'(t)| \leq \varepsilon$$

by (2). Hence for $m, n > k$, $x \in J$ and

$$|F_m(x) - F_n(x) - F_m(p) + F_n(p)| \leq \varepsilon|x - p| \leq \varepsilon\delta. \tag{3}$$

As ε is *arbitrary*, this shows that the sequence

$$\{F_n - F_n(p)\}$$

satisfies the uniform Cauchy criterion (Chapter 4, §12, Theorem 3). Thus as E is complete, $\{F_n - F_n(p)\}$ converges uniformly on J. So does $\{F_n\}$, for $\{F_n(p)\}$ *converges*, by (a). Thus we may set

$$F = \lim F_n \text{ (uniformly) on } J,$$

proving assertion (i).[1]

Here by Theorem 2 of Chapter 4, §12, F is relatively continuous on *each* such $J \subseteq I$, hence on all of I. Also, letting $m \to +\infty$ (with n fixed), we have $F_m \to F$ in (3), and it follows that for $n > k$ and $x \in G_p(\delta) \cap I$.

$$|F(x) - F_n(x) - F(p) + F_n(p)| \le \varepsilon|x - p| \le \varepsilon\delta. \tag{4}$$

Having proved (i), we may now treat p as just *any* point in I. Thus formula (4) holds for any globe $G_p(\delta)$, $p \in I$. We now show that

$$F' = f \text{ on } I - Q; \text{ i.e., } F = \int f \text{ on } I.$$

Indeed, if $p \in I - Q$, each F_n is differentiable at p (by assumption), and $p \in I^0$ (since $I - Q \subseteq I^0$ by our convention). Thus for each n, there is $\delta_n > 0$ such that

$$\left|\frac{\Delta F_n}{\Delta x} - F_n'(p)\right| = \left|\frac{F_n(x) - F_n(p)}{x - p} - F_n'(p)\right| < \frac{\varepsilon}{2} \tag{5}$$

for all $x \in G_{\neg p}(\delta_n)$; $G_p(\delta_n) \subseteq I$.

By assumption (b) and by (4), we can fix n so large that

$$|F_n'(p) - f(p)| < \frac{\varepsilon}{2}$$

and so that (4) holds for $\delta = \delta_n$. Then, dividing by $|\Delta x| = |x - p|$, we have

$$\left|\frac{\Delta F}{\Delta x} - \frac{\Delta F_n}{\Delta x}\right| \le \varepsilon.$$

Combining with (5), we infer that for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left|\frac{\Delta F}{\Delta x} - f(p)\right| \le \left|\frac{\Delta F}{\Delta x} - \frac{\Delta F_n}{\Delta x}\right| + \left|\frac{\Delta F_n}{\Delta x} - F_n'(p)\right| + |F_n'(p) - f(p)| < 2\varepsilon,\ x \in G_p(\delta).$$

This shows that

$$\lim_{x \to p} \frac{\Delta F}{\Delta x} = f(p) \text{ for } p \in I - Q,$$

i.e., $F' = f$ on $I - Q$, with f *finite* by assumption, and F finite by (4). As F is also relatively continuous on I, assertion (ii) is proved, and (iii) follows since $F = \lim F_n$ and $f = \lim F_n'$. □

Note 1. The same proof also shows that $F_n \to F$ (uniformly) on each *closed* subinterval $J \subseteq I$ if $F_n' \to f$ (uniformly) on $J - Q$ for all *such* J (with the other assumptions unchanged). In any case, we then have $F_n \to F$ (pointwise) on all of I.

We now apply Theorem 1 to *antiderivatives*.

[1] Indeed, any J can be enlarged to include p, so (3) applies to it. Note that in (3) we may as well vary x inside any set of the form $I \cap G_p(\delta)$.

Theorem 2. *Let the functions $f_n\colon E^1 \to E$, $n = 1, 2, \ldots$, have antiderivatives, $F_n = \int f_n$, on I. Suppose E is complete and $f_n \to f$ (uniformly) on each finite subinterval $J \subseteq I$, with f finite there. Then $\int f$ exists on I, and*

$$\int_p^x f = \int_p^x \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_p^x f_n \text{ for any } p,\, x \in I. \tag{6}$$

Proof. Fix any $p \in I$. By Note 2 in §5, we may choose

$$F_n(x) = \int_p^x f_n \text{ for } x \in I.$$

Then $F_n(p) = \int_p^p f_n = 0$, and so $\lim_{n\to\infty} F_n(p) = 0$ exists, as required in Theorem 1(a).

Also, by definition, each F_n is relatively continuous and finite on I and differentiable, with $F_n' = f_n$, on $I - Q_n$. The countable sets Q_n need not be the same, so we replace them by

$$Q = \bigcup_{n=1}^{\infty} Q_n$$

(including in Q also the endpoints of I, if any). Then Q *is countable* (see Chapter 1, §9, Theorem 2), and $I - Q \subseteq I - Q_n$, so *all* F_n are differentiable on $I - Q$, with $F_n' = f_n$ there.

Additionally, by assumption,

$$f_n \to f \ (uniformly)$$

on finite subintervals $J \subseteq I$. Hence

$$F_n' \to f \ (uniformly) \text{ on } J - Q$$

for all such J, and so the conditions of Theorem 1 are satisfied.

By that theorem, then,

$$F = \int f = \lim F_n \text{ exists on } I$$

and, recalling that

$$F_n(x) = \int_p^x f_n,$$

we obtain for $x \in I$

$$\int_p^x f = F(x) - F(p) = \lim F_n(x) - \lim F_n(p) = \lim F_n(x) - 0 = \lim \int_p^x f_n.$$

As $p \in I$ was arbitrary, and $f = \lim f_n$ (by assumption), all is proved. □

Note 2. By Theorem 1, the convergence

$$\int_p^x f_n \to \int_p^x f \quad \text{(i.e., } F_n \to F\text{)}$$

is *uniform* in x (with p fixed), on each finite subinterval $J \subseteq I$.

We now "translate" Theorems 1 and 2 into the language of series.

Corollary 1. *Let E and the functions $F_n\colon E^1 \to E$ be as in Theorem 1. Suppose the series*

$$\sum F_n(p)$$

converges for some $p \in I$, and

$$\sum F_n'$$

converges uniformly on $J - Q$, for each finite subinterval $J \subseteq I$.

Then $\sum F_n$ converges uniformly on each such J, and

$$F = \sum_{n=1}^{\infty} F_n$$

is differentiable on $I - Q$, with

$$F' = \left(\sum_{n=1}^{\infty} F_n\right)' = \sum_{n-1}^{\infty} F_n' \text{ there.} \tag{7}$$

In other words, the series can be differentiated *termwise*.

Proof. Let

$$s_n = \sum_{k=1}^{n} F_k, \quad n = 1, 2, \ldots,$$

be the partial sums of $\sum F_n$. From our assumptions, it then follows that the s_n satisfy all conditions of Theorem 1. (Verify!) Thus the conclusions of Theorem 1 hold, with F_n replaced by s_n.

We have $F = \lim s_n$ and $F' = (\lim s_n)' = \lim s_n'$, whence (7) follows. □

Corollary 2. *If E and the f_n are as in Theorem 2 and if $\sum f_n$ converges uniformly to f on each finite interval $J \subseteq I$, then $\int f$ exists on I, and*

$$\int_p^x f = \int_p^x \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_p^x f_n \text{ for any } p,\, x \in I. \tag{8}$$

Briefly, a uniformly convergent series can be integrated *termwise*.

Theorem 3 (power series)**.** *Let r be the convergence radius of*

$$\sum a_n(x-p)^n, \quad a_n \in E,\, p \in E^1.$$

Suppose E is complete. Set

$$f(x) = \sum_{n=0}^{\infty} a_n(x-p)^n \quad on\ I = (p-r,\, p+r).$$

Then the following are true:

(i) *f is differentiable and has an exact antiderivative on I.*

(ii) $f'(x) = \sum_{n=1}^{\infty} na_n(x-p)^{n-1}$ *and* $\int_p^x f = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-p)^{n+1}$, $x \in I$.

(iii) *r is also the convergence radius of the two series in* (ii).

(iv) $\sum_{n=0}^{\infty} a_n(x-p)^n$ *is exactly the Taylor series for $f(x)$ on I about p.*

Proof. We prove (iii) first.

By Theorem 6 of Chapter 4, §13, $r = 1/d$, where

$$d = \overline{\lim}\ \sqrt[n]{a_n}.$$

Let r' be the convergence radius of $\sum na_n(x-p)^{n-1}$, so

$$r' = \frac{1}{d'} \text{ with } d' = \overline{\lim}\ \sqrt[n]{na_n}.$$

However, $\lim \sqrt[n]{n} = 1$ (see §3, Example (e)). It easily follows that

$$d' = \overline{\lim}\ \sqrt[n]{na_n} = 1 \cdot \overline{\lim}\ \sqrt[n]{a_n} = d.^2$$

Hence $r' = 1/d' = 1/d = r$.

The convergence radius of $\sum \frac{a_n}{n+1}(x-p)^{n+1}$ is determined similarly. Thus (iii) is proved.

Next, let

$$f_n(x) = a_n(x-p)^n \text{ and } F_n(x) = \frac{a_n}{n+1}(x-p)^{n+1},\ n = 0, 1, 2, \ldots.$$

Then the F_n are differentiable on I, with $F_n' = f_n$ there. Also, by Theorems 6 and 7 of Chapter 4, §13, the series

$$\sum F_n' = \sum a_n(x-p)^n$$

[2] For a proof, treat d and d' as *subsequential* limits (Chapter 4, §16, Theorem 1; Chapter 2, §13, Problem 4).

converges uniformly on each closed subinterval $J \subseteq I = (p - r, p + r)$.[3] Thus the functions F_n satisfy all conditions of Corollary 1, with $Q = \emptyset$, and the f_n satisfy Corollary 2. By Corollary 1, then,

$$F = \sum_{n=1}^{\infty} F_n$$

is differentiable on I, with

$$F'(x) = \sum_{n=1}^{\infty} F_n'(x) = \sum_{n=1}^{\infty} a_n (x - p)^n = f(x)$$

for all $x \in I$. Hence F is an *exact* antiderivative of f on I, and (8) yields the *second* formula in (ii).

Quite similarly, replacing F_n and F by f_n and f, one shows that f is differentiable on I, and the *first* formula in (ii) follows. This proves (i) as well.

Finally, to prove (iv), we apply (i)–(iii) to the consecutive derivatives of f and obtain

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n (x-p)^{n-k}$$

for each $x \in I$ and $k \in N$.

If $x = p$, all terms vanish except the *first* term ($n = k$), i.e., $k!\, a_k$. Thus $f^{(k)}(p) = k!\, a_k$, $k \in N$. We may rewrite it as

$$a_n = \frac{f^{(n)}(p)}{n!}, \quad n = 0, 1, 2, \ldots,$$

since $f^{(0)}(p) = f(p) = a_0$. Assertion (iv) now follows since

$$f(x) = \sum_{n=0}^{\infty} a_n (x-p)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (x-p)^n, \quad x \in I, \text{ as required.} \quad \square$$

Note 3. If $\sum a_n (x-p)^n$ converges also for $x = p - r$ or $x = p + r$, so does the *integrated* series

$$\sum a_n \frac{(x-p)^{n+1}}{n+1}$$

since we may include such x in I. However, the *derived* series $\sum n a_n (x-p)^{n-1}$ *need not* converge at such x. (Why?) For example (see §6, Problem 9), the expansion

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

[3] For our present theorem, it suffices to show that it holds on any *closed globe* $J = \overline{G}_p(\delta)$, $\delta < r$. We may therefore limit ourselves to such J (see Note 1).

converges for $x = 1$ but the *derived* series

$$1 - x + x^2 - \cdots$$

does not.

In this respect, there is the following useful rule for functions $f\colon E^1 \to E^m$ $({}^*C^m)$.

Corollary 3. *Let a function $f\colon E^1 \to E^m$ $({}^*C^m)$ be relatively continuous on $[p, x_0]$ (or $[x_0, p]$), $x_0 \neq p$.*[4] *If*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-p)^n \textit{ for } p \leq x < x_0 \ (\textit{respectively, } x_0 < x \leq p),$$

and if $\sum a_n (x_0 - p)^n$ *converges, then necessarily*

$$f(x_0) = \sum_{n=0}^{\infty} a_n (x_0 - p)^n.$$

The proof is sketched in Problems 4 and 5.

Thus in the above example, $f(x) = \ln(1+x)$ defines a *continuous* function on $[0, 1]$, with

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \text{ on } [0, 1].$$

We gave a direct proof in §6, Problem 9. However, by Corollary 3, it suffices to prove this for $[0, 1)$, which is much easier. Then the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

yields the result for $x = 1$ as well.

Problems on Convergence in Differentiation and Integration

1. Complete all proof details in Theorems 1 and 3, Corollaries 1 and 2, and Note 3.

2. Show that assumptions (a) and (c) in Theorem 1 can be replaced by $F_n \to F$ (pointwise) *on* I. (In this form, the theorem applies to *incomplete* spaces E as well.)
   [Hint: $F_n \to F$ (pointwise), together with formula (3), *implies* $F_n \to F$ (uniformly) on I.]

[4] Relative continuity at x_0 suffices.

3. Show that Theorem 1 *fails* without assumption (b), even if $F_n \to F$ (*uniformly*) and if F is differentiable on I.
[Hint: For a counterexample, try $F_n(x) = \frac{1}{n} \sin nx$, on *any* nondegenerate I. Verify that $F_n \to 0$ (uniformly), yet (b) and assertion (iii) fail.]

4. Prove *Abel's theorem* (Chapter 4, §13, Problem 15) for series

$$\sum a_n (x-p)^n,$$

with all a_n in E^m (*or in C^m) but with $x, p \in E^1$.
[Hint: Split $a_n(x-p)^n$ into *components*.]

5. Prove Corollary 3.
[Hint: By Abel's theorem (see Problem 4), we may put

$$\sum_{n=0}^{\infty} a_n (x-p)^n = F(x)$$

uniformly on $[p, x_0]$ (respectively, $[x_0, p]$). This implies that F is relatively continuous at x_0. (Why?) So is f, by assumption. Also $f = F$ on $[p, x_0)$ ($(x_0, p]$). Show that

$$f(x_0) = \lim f(x) = \lim F(x) = F(x_0)$$

as $x \to x_0$ from the left (right).]

6. In the following cases, find the Taylor series of F about 0 by integrating the series of F'. Use Theorem 3 and Corollary 3 to find the convergence radius r and to investigate convergence at $-r$ and r. Use (b) to find a formula for π.

(a) $F(x) = \ln(1+x)$;

(b) $F(x) = \arctan x$;

(c) $F(x) = \arcsin x$.

7. Prove that

$$\int_0^x \frac{\ln(1-t)}{t}\, dt = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \quad \text{for } x \in [-1, 1].$$

[Hint: Use Theorem 3 and Corollary 3. Take derivatives of both sides.]

§10. Sufficient Condition of Integrability. Regulated Functions

In this section, we shall determine a large family of functions that do have antiderivatives. First, we give a general definition, valid for *any* range space (T, p) (not necessarily E). The domain space remains E^1.

Definition 1.

A function $f\colon E^1 \to (T, p)$ is said to be *regulated* on an interval $I \subseteq E^1$, with endpoints $a < b$, iff the limits $f(p^-)$ and $f(p^+)$, *other than* $\pm\infty$,[1] exist at each $p \in I$. However, at the endpoints a, b, if *in* I, we only require $f(a^+)$ and $f(b^-)$ to exist.

Examples.

(a) If f is relatively continuous and finite on I, it is regulated.

(b) Every real monotone function is regulated (see Chapter 4, §5, Theorem 1).

(c) If $f\colon E^1 \to E^n$ $({}^*C^n)$ has bounded variation on I, it is regulated (§7, Theorem 4).[2]

(d) The *characteristic* function of a set B, denoted C_B, is defined by

$$C_B(x) = 1 \text{ if } x \in B \text{ and } C_B = 0 \text{ on } -B.$$

For any interval $J \subseteq E^1$, C_J *is regulated on* E^1.

(e) A function f is called a *step function* on I iff I can be represented as the union, $I = \bigcup_k I_k$, of a sequence of disjoint intervals I_k such that f is constant and $\neq \pm\infty$ on each I_k. Note that some I_k may be *singletons*, $\{p\}$.[3]

If the number of the I_k is finite, we call f a *simple step function.*

When the range space T is E, we can give the following convenient alternative definition. If, say, $f = a_k \neq \pm\infty$ on I_k, then

$$f = \sum_k a_k C_{I_k} \quad \text{on } I,$$

where C_{I_k} is as in (d). Note that $\sum_k a_k C_{I_k}(x)$ always exists for disjoint I_k. (Why?)

Each simple step function is regulated. (Why?)

Theorem 1. *Let the functions f, g, h be real or complex (or let f, g be vector valued and h scalar valued).*

If they are regulated on I, so are $f \pm g$, fh, and $|f|$; so also is f/h if h is bounded away from 0 *on I, i.e.,* $(\exists\, \varepsilon > 0)$ $|h| \geq \varepsilon$ *on I.*

The proof, based on the usual limit properties, is left to the reader.

We shall need two lemmas. One is the famous *Heine–Borel* lemma.

[1] This restriction is necessary in *integration* only, in the case $T = E^1$ or $T = E^*$.

[2] Actually, this applies to *any* $f\colon E^1 \to E$, with E *complete* and $V_f[I] < +\infty$ (Problem 7).

[3] The endpoints of the I_k may be treated as such degenerate intervals.

Lemma 1 (Heine–Borel)**.** *If a closed interval $A = [a, b]$ in E^1 (or E^n) is covered by open sets G_i $(i \in I)$, i.e.,*

$$A \subseteq \bigcup_{i \in I} G_i,$$

then A can be covered by a finite *number of these G_i.*

The proof was sketched in Problem 10 of Chapter 4, §6.

Note 1. This fails for nonclosed intervals A. For example, let

$$A = (0, 1) \subseteq E^1 \text{ and } G_n = \left(\frac{1}{n}, 1\right).$$

Then

$$A = \bigcup_{n=1}^{\infty} G_n \text{ (verify!), but not } A \subseteq \bigcup_{n=1}^{m} G_n$$

for any *finite* m. (Why?)

The lemma also fails for nonopen sets G_i. For example, cover A by *singletons* $\{x\}$, $x \in A$. Then *none* of the $\{x\}$ can be dropped!

Lemma 2. *If a function $f\colon E^1 \to T$ is regulated on $I = [a, b]$, then f can be uniformly approximated by simple step functions on I.*

That is, for any $\varepsilon > 0$, there is a simple step function g, with $\rho(f, g) \leq \varepsilon$ on I; hence

$$\sup_{x \in I} \rho(f(x), g(x)) \leq \varepsilon.$$

Proof. By assumption, $f(p^-)$ exists for each $p \in (a, b]$, and $f(p^+)$ exists for $p \in [a, b)$, all finite.

Thus, given $\varepsilon > 0$ and any $p \in I$, there is $G_p(\delta)$ (δ depending on p) such that $\rho(f(x), r) < \varepsilon$ whenever $r = f(p^-)$ and $x \in (p - \delta, p)$, and $\rho(f(x), s) < \varepsilon$ whenever $s = f(p^+)$ and $x \in (p, p + \delta)$; $x \in I$.

We choose such a $G_p(\delta)$ for *every* $p \in I$. Then *the open globes $G_p = G_p(\delta)$ cover the closed interval $I = [a, b]$*, so by Lemma 1, I is covered by a *finite* number of such globes, say,

$$I \subseteq \bigcup_{k=1}^{n} G_{p_k}(\delta_k), \quad a \in G_{p_1},\ a \leq p_1 < p_2 < \cdots < p_n \leq b.$$

We now define the step function g on I as follows.

If $x = p_k$, we put

$$g(x) = f(p_k), \quad k = 1, 2, \ldots, n.$$

If $x \in [a, p_1)$, then

$$g(x) = f(p_1^-).$$

If $x \in (p_1, p_1 + \delta_1)$, then
$$g(x) = f(p_1^+).$$
More generally, if x is in $G_{\neg p_k}(\delta_k)$ but in none of the $G_{p_i}(\delta_i)$, $i < k$, we put
$$g(x) = f(p_k^-) \quad \text{if } x < p_k$$
and
$$g(x) = f(p_k^+) \quad \text{if } x > p_k.$$
Then by construction, $\rho(f, g) < \varepsilon$ on each G_{p_k}, hence on I. $\square$

***Note 2.** If T is *complete*, we can say more: f is regulated on $I = [a, b]$ *iff* f is uniformly approximated by simple step functions on I. (See Problem 2.)

Theorem 2. *If $f\colon E^1 \to E$ is regulated on an interval $I \subseteq E^1$ and if E is complete, then $\int f$ exists on I, exact at every continuity point of f in I^0.*

In particular, all continuous maps $f\colon E^1 \to E^n$ ($^*C^n$) have exact primitives.

Proof. In view of Problem 14 of §5, it suffices to consider *closed* intervals.

Thus let $I = [a, b]$, $a < b$, in E^1. Suppose first that f is the characteristic function C_J of a subinterval $J \subseteq I$ with endpoints c and d ($a \leq c \leq d \leq b$), so $f = 1$ on J, and $f = 0$ on $I - J$. We then define $F(x) = x$ on J, $F = c$ on $[a, c]$, and $F = d$ on $[d, b]$ (see Figure 25). Thus F is continuous (why?), and $F' = f$ on $I - \{a, b, c, d\}$ (why?). Hence $F = \int f$ on I; i.e., *characteristic functions are integrable.*

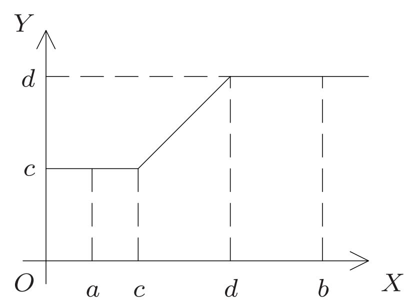

FIGURE 25

Then, however, so is any simple step function
$$f = \sum_{k=1}^{m} a_k C_{I_k},$$
by repeated use of Corollary 1 in §5.[4]

Finally, let f be *any* regulated function on I. Then by Lemma 2, for any $\varepsilon_n = \frac{1}{n}$, there is a simple step function g_n such that
$$\sup_{x \in I} |g_n(x) - f(x)| \leq \frac{1}{n}, \quad n = 1, 2, \ldots.$$
As $\frac{1}{n} \to 0$, this implies that $g_n \to f$ (uniformly) on I (see Chapter 4, §12, Theorem 1). Also, by what was proved above, the step functions g_n have

[4] The corollary applies here also if the a_k are *vectors* (C_{I_k} is *scalar valued*).

antiderivatives, *hence so has* f (Theorem 2 in §9); i.e., $F = \int f$ exists on I, as claimed. Moreover, $\int f$ is exact at continuity points of f in I^0 (Problem 10 in §5). □

In view of the sufficient condition expressed in Theorem 2, we can now replace the assumption "$\int f$ exists" in our previous theorems by "f is regulated" (provided E is complete). For example, let us now review Problems 7 and 8 in §5.

Theorem 3 (weighted law of the mean)**.** *Let $f\colon E^1 \to E$ (E complete) and $g\colon E^1 \to E^1$ be regulated on $I = [a, b]$, with $g \geq 0$ on I.*[5] *Then the following are true:*

(i) *There is a finite $c \in E$ (called the "g-*weighted *mean of f on I") such that $\int_a^b gf = c\int_a^b g$.*

(ii) *If f, too, is real and has the Darboux property on I, then $c = f(q)$ for some $q \in I$.*

Proof. Indeed, as f and g are regulated, so is gf by Theorem 1. Hence by Theorem 2, $\int f$ and $\int gf$ exist on I. The rest follows as in Problems 7 and 8 of §5. □

Theorem 4 (second law of the mean)**.** *Suppose f and g are real, f is monotone with $f = \int f'$ on I, and g is regulated on I; $I = [a, b]$. Then*

$$\int_a^b fg = f(a)\int_a^q g + f(b)\int_q^b g \text{ for some } q \in I. \tag{1}$$

Proof. To fix ideas, let $f\uparrow$; i.e., $f' \geq 0$ on I.

The formula $f = \int f'$ means that f is relatively continuous (hence regulated) on I and differentiable on $I - Q$ (Q countable). As g is regulated,

$$\int_a^x g = G(x)$$

does exist on I, so G has similar properties, with $G(a) = \int_a^a g = 0$.

By Theorems 1 and 2, $\int fG' = \int fg$ exists on I. (Why?) Hence by Corollary 5 in §5, so does $\int Gf'$, and we have

$$\int_a^b fg = \int_a^b fG' = f(x)G(x)\Big|_a^b - \int_a^b Gf' = f(b)G(b) - \int_a^b Gf'.$$

Now G has the Darboux property on I (being relatively continuous), and

[5] One can also assume $g \leq 0$ on I; in this case, simply apply the theorem to $-g$.

$f' \geq 0$. Also, $\int G$ and $\int Gf'$ exist on I. Thus by Problems 7 and 8 in §5,

$$\int_a^b Gf' = G(q)\int_a^b f' = G(q)f(x)\Big|_a^b, \quad q \in I.$$

Combining all, we obtain the required result (1) since

$$\begin{aligned}\int fg &= f(b)G(b) - \int_a^b Gf' \\ &= f(b)G(b) - f(b)G(q) + f(a)G(q) \\ &= f(b)\int_q^b g + f(a)\int_a^q g. \quad \square\end{aligned}$$

We conclude with an application to infinite series. Given $f\colon E^1 \to E$, we define

$$\int_a^\infty f = \lim_{x\to+\infty}\int_a^x f \text{ and } \int_{-\infty}^a f = \lim_{x\to-\infty}\int_x^a f$$

if these integrals and limits exist.

We say that $\int_a^\infty f$ and $\int_{-\infty}^a f$ *converge* iff they exist and are *finite*.

Theorem 5 (integral test of convergence)**.** *If $f\colon E^1 \to E^1$ is nonnegative and nonincreasing on $I = [a, +\infty)$, then*

$$\int_a^\infty f \textit{ converges iff } \sum_{n=1}^\infty f(n) \textit{ does.}$$

Proof. As $f\!\downarrow$, f is regulated, so $\int f$ exists on $I = [a, +\infty)$. We fix some natural $k \geq a$ and define

$$F(x) = \int_k^x f \text{ for } x \in I.$$

By Theorem 3(iii) in §5, $F\!\uparrow$ on I. Thus by monotonicity,

$$\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty}\int_k^x f = \int_k^\infty f$$

exists *in* E^*; so does $\int_a^k f$. Since

$$\int_a^x f = \int_a^k f + \int_k^x f,$$

where $\int_a^k f$ is *finite* by definition, we have

$$\int_a^\infty f < +\infty \quad \text{iff} \quad \int_k^\infty f < +\infty.$$

Similarly,

$$\sum_{n=1}^{\infty} f(n) < +\infty \quad \text{iff} \quad \sum_{n=k}^{\infty} f(n) < +\infty.$$

Thus we may replace "a" by "k."

Let

$$I_n = [n,\, n+1), \quad n = k,\, k+1,\, \ldots,$$

and define two step functions, g and h, constant on each I_n, by

$$h = f(n) \text{ and } g = f(n+1) \text{ on } I_n,\, n \geq k.$$

Since $f\downarrow$, we have $g \leq f \leq h$ on all I_n, hence on $J = [k, +\infty)$. Therefore,

$$\int_k^x g \leq \int_k^x f \leq \int_k^x h \text{ for } x \in J.$$

Also,

$$\int_k^m h = \sum_{n=k}^{m-1} \int_n^{n+1} h = \sum_{n=k}^{m-1} f(n),$$

since $h = f(n)$ (constant) on $[n,\, n+1)$, and so

$$\int_n^{n+1} h(x)\, dx = f(n) \int_n^{n+1} 1\, dx = f(n) \cdot x \Big|_n^{n+1} = f(n)(n+1-n) = f(n).$$

Similarly,

$$\int_k^m g = \sum_{n=k}^{m-1} f(n+1) = \sum_{n=k+1}^{m} f(n).$$

Thus we obtain

$$\sum_{n=k+1}^{m} f(n) = \int_k^m g \leq \int_k^m f \leq \int_k^m h = \sum_{n=k}^{m-1} f(n),$$

or, letting $m \to \infty$,

$$\sum_{n=k+1}^{\infty} f(n) \leq \int_k^{\infty} f \leq \sum_{n=k}^{\infty} f(n).$$

Hence $\int_k^{\infty} f$ is finite iff $\displaystyle\sum_{n=1}^{\infty} f(n)$ is, and all is proved. □

Examples (continued)**.**

(f) Consider the hyperharmonic series

$$\sum \frac{1}{n^p} \quad \text{(Problem 2 of Chapter 4, §13).}$$

Let

$$f(x) = \frac{1}{x^p}, \quad x \geq 1.$$

If $p = 1$, then $f(x) = 1/x$, so $\int_1^x f = \ln x \to +\infty$ as $x \to +\infty$. Hence $\sum 1/n$ diverges.

If $p \neq 1$, then

$$\int_1^\infty f = \lim_{x \to +\infty} \int_1^x f = \lim_{x \to +\infty} \frac{x^{1-p}}{1-p}\bigg|_1^x,$$

so $\int_1^\infty f$ converges or diverges according as $p > 1$ or $p < 1$, and the same applies to the series $\sum 1/n^p$.

(g) *Even nonregulated functions may be integrable.* Such is *Dirichlet's function* (Example (c) in Chapter 4, §1). Explain, using the countability of the rationals.

Problems on Regulated Functions

In Problems 2, 5, 6, and 8, we drop the restriction that $f(p^-)$ and $f(p^+)$ are *finite*. We only require them to exist *in* (T, p). If $T = E^*$, a suitable metric for E^* is presupposed.

1. Complete all details in the proof of Theorems 1–3.

1′ Explain Examples (a)–(g).

***2.** Prove Note 2. More generally, assuming T to be complete, prove that if

$$g_n \to f \text{ (uniformly) on } I = [a, b]$$

and if the g_n are regulated on I, so is f.

[Hint: Fix $p \in (a, b]$. Use Theorem 2 of Chapter 4, §11 with

$$X = [a, p],\ Y = N \cup \{+\infty\},\ q = +\infty, \text{ and } F(x, n) = g_n(x).$$

Thus show that

$$f(p^-) = \lim_{x \to p^-} \lim_{n \to \infty} g_n(x) \text{ exists;}$$

similarly for $f(p^+)$.]

3. Given $f, g\colon E^1 \to E^1$, define $f \vee g$ and $f \wedge g$ as in Problem 12 of Chapter 4, §8. Using the hint given there, show that $f \vee g$ and $f \wedge g$ are regulated if f and g are.

4. Show that the function $g \circ f$ need not be regulated even if g and f are.
[Hint: Let

$$f(x) = x \cdot \sin \frac{1}{x},\ g(x) = \frac{x}{|x|},\ \text{and } f(0) = g(0) = 0 \text{ with } I = [0,\,1].$$

Proceed.]

⇒**5.** Given $f\colon E^1 \to (T,\,\rho)$, regulated on I, put

$$j(p) = \max\{\rho\big(f(p),\, f(p^-)\big),\, \rho\big(f(p),\, f(p^+)\big),\, \rho\big(f(p^-),\, f(p^+)\big)\};$$

call it the *jump* at p.

(i) Prove that f is discontinuous at $p \in I^0$ iff $j(p) > 0$, i.e., iff

$$(\exists\, n \in N) \quad j(p) > \frac{1}{n}.$$

(ii) For a fixed $n \in N$, prove that a closed subinterval $J \subseteq I$ contains at most finitely many x with $j(x) > 1/n$.
[Hint: Otherwise, there is a sequence of distinct points $x_m \in J$, $j(x_m) > \frac{1}{n}$, hence a subsequence $x_{m_k} \to p \in J$. (Why?) Use Theorem 1 of Chapter 4, §2, to show that $f(p^-)$ or $f(p^+)$ *fails* to exist.]

⇒**6.** Show that if $f\colon E^1 \to (T,\,\rho)$ is regulated on I, then it has at most countably many discontinuities in I; all are of the "jump" type (Problem 5).
[Hint: By Problem 5, any closed subinterval $J \subseteq I$ contains, for each n, at most *finitely many* discontinuities x with $j(x) > 1/n$. Thus for $n = 1, 2, \ldots,$ obtain *countably* many such x.]

7. Prove that if E is complete, all maps $f\colon E^1 \to E$, with $V_f[I] < +\infty$ on $I = [a,\,b]$, are regulated on I.
[Hint: Use Corollary 1, Chapter 4, §2, to show that $f(p^-)$ and $f(p^+)$ exist.
Say,

$$x_n \to p \text{ with } x_n < p \quad (x_n,\, p \in I),$$

but $\{f(x_n)\}$ is *not* Cauchy. Then find a subsequence, $\{x_{n_k}\}\!\uparrow$, and $\varepsilon > 0$ such that

$$|f(x_{n_{k+1}}) - f(x_{n_k})\,| \geq \varepsilon, \quad k = 1, 3, 5, \ldots.$$

Deduce a contradiction to $V_f[I] < +\infty$.
Provide a similar argument for the case $x_n > p$.]

8. Prove that if $f\colon E^1 \to (T,\,\rho)$ is regulated on I, then $\overline{f[B]}$ (the closure of $f[B]$) is compact in $(T,\,\rho)$ whenever B is a compact subset of I.
[Hint: Given $\{z_m\}$ in $\overline{f[B]}$, find $\{y_m\} \subseteq f[B]$ such that $\rho(z_m,\, y_m) \to 0$ (use Theorem 3 of Chapter 3, §16). Then "imitate" the proof of Theorem 1 in Chapter 4, §8 suitably. Distinguish the cases:

(i) all but finitely many x_m are $< p$;

(ii) infinitely many x_m exceed p; or

(iii) infinitely many x_m equal p.]

§11. Integral Definitions of Some Functions

By Theorem 2 in §10, $\int f$ exists on I whenever the function $f\colon E^1 \to E$ is regulated on I, and E is complete. Hence whenever such an f is given, we can define a new function F by setting

$$F = \int_a^x f$$

on I for some $a \in I$. This is a convenient method of obtaining new continuous functions, differentiable on $I - Q$ (Q countable). We shall now apply it to obtain new definitions of some functions previously defined in a rather strenuous step-by-step manner.

I. Logarithmic and Exponential Functions. From our former definitions, we *proved* that

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

Now we want to treat this as a *definition* of logarithms. We start by setting

$$f(t) = \frac{1}{t}, \quad t \in E^1,\ t \neq 0,$$

and $f(0) = 0$.

Then f is continuous on $I = (0, +\infty)$ and $J = (-\infty, 0)$, so it has an *exact* primitive on I and J *separately* (not on E^1). Thus we can now *define* the log function on I by

$$\int_1^x \frac{1}{t}\,dt = \log x \text{ (also written } \ln x\text{) for } x > 0. \tag{1}$$

By the very definition of an exact primitive, the log function is continuous and differentiable on $I = (0, +\infty)$; its derivative on I is f. Thus we again have the symbolic formula

$$(\log x)' = \frac{1}{x}, \quad x > 0.$$

If $x < 0$, we can consider $\log(-x)$. Then the chain rule (Theorem 3 of §1) yields

$$(\log(-x))' = \frac{1}{x}. \quad \text{(Verify!)}$$

Hence

$$(\log|x|)' = \frac{1}{x} \quad \text{for } x \neq 0. \tag{2}$$

Other properties of logarithms easily follow from (1). We summarize them now.

Theorem 1.

(i) $\log 1 = \int_1^1 \frac{1}{t}\, dt = 0.$

(ii) $\log x < \log y$ *whenever* $0 < x < y$.

(iii) $\lim_{x \to +\infty} \log x = +\infty$ *and* $\lim_{x \to 0^+} \log x = -\infty$.

(iv) *The range of* $\log$ *is all of* E^1.

(v) *For any positive* $x,\ y \in E^1$,

$$\log(xy) = \log x + \log y \ \textit{and} \ \log\left(\frac{x}{y}\right) = \log x - \log y.$$

(vi) $\log a^r = r \cdot \log a,\ a > 0,\ r \in N.$

(vii) $\log e = 1,$ *where* $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$

Proof.

(ii) By (2), $(\log x)' > 0$ on $I = (0, +\infty)$, so $\log x$ is increasing on I.

(iii) By Theorem 5 in §10,

$$\lim_{x \to +\infty} \log x = \int_1^\infty \frac{1}{t}\, dt = +\infty$$

since

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty \quad \text{(Chapter 4, §13, Example (b))}.$$

Hence, substituting $y = 1/x$, we obtain

$$\lim_{y \to 0^+} \log y = \lim_{x \to +\infty} \log \frac{1}{x}.$$

However, by Theorem 2 in §5 (substituting $s = 1/t$),

$$\log \frac{1}{x} = \int_1^{1/x} \frac{1}{t}\, dt = -\int_1^x \frac{1}{s}\, ds = -\log x.$$

Thus

$$\lim_{y \to 0^+} \log y = \lim_{x \to +\infty} \log \frac{1}{x} = -\lim_{x \to +\infty} \log x = -\infty$$

as claimed. (We also proved that $\log \frac{1}{x} = -\log x$.)

(iv) Assertion (iv) now follows by the Darboux property (as in Chapter 4, §9, Example (b)).

(v) With x, y fixed, we substitute $t = xs$ in

$$\int_1^{xy} \frac{1}{t}\,dt = \log xy$$

and obtain

$$\begin{aligned}\log xy &= \int_1^{xy} \frac{1}{t}\,dt = \int_{1/x}^{y} \frac{1}{s}\,ds \\ &= \int_{1/x}^{1} \frac{1}{s}\,ds + \int_1^{y} \frac{1}{s}\,ds \\ &= -\log\frac{1}{x} + \log y \\ &= \log x + \log y.\end{aligned}$$

Replacing y by $1/y$ here, we have

$$\log\frac{x}{y} = \log x + \log\frac{1}{y} = \log x - \log y.$$

Thus (v) is proved, and (vi) follows by induction over r.

(vii) By continuity,

$$\log e = \lim_{x\to e} \log x = \lim_{n\to\infty} \log\Bigl(1 + \frac{1}{n}\Bigr)^n = \lim_{n\to\infty} \frac{\log(1 + 1/n)}{1/n},$$

where the last equality follows by (vi). Now, L'Hôpital's rule yields

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{1}{1+x} = 1.$$

Letting x run over $\frac{1}{n} \to 0$, we get (vii). □

Note 1. Actually, (vi) holds for *any* $r \in E^1$, with a^r as in Chapter 2, §§11–12. One uses the techniques from that section to prove it first for rational r, and then it follows for all real r by continuity. However, we prefer not to use this now.

Next, we define the *exponential function* ("exp") to be the inverse of the log function. This inverse function exists; it is continuous (even differentiable) and strictly increasing on its domain (by Theorem 3 of Chapter 4, §9 and Theorem 3 of Chapter 5, §2) since the log function has these properties. From $(\log x)' = 1/x$ we get, as in §2,

$$(\exp x)' = \exp x \quad \text{(cf. §2, Example (B))}. \tag{3}$$

The domain of the exponential is the *range* of its inverse, i.e., E^1 (cf. Theorem 1(iv)). Thus $\exp x$ is defined for all $x \in E^1$. The range of exp is the domain of log, i.e., $(0, +\infty)$. Hence $\exp x > 0$ for all $x \in E^1$. Also, by definition,

$$\exp(\log x) = x \text{ for } x > 0, \tag{4}$$

$$\exp 0 = 1 \text{ (cf. Theorem 1(i)), and} \tag{5}$$

$$\exp r = e^r \text{ for } r \in N. \tag{6}$$

Indeed, by Theorem 1(vi) and (vii), $\log e^r = r \cdot \log e = r$. Hence (6) follows. If the definitions and rules of Chapter 2, §§11–12 are used, this proof even works for *any* r by Note 1. Thus *our new definition of* exp *agrees with the old one*.

Our next step is to give a new definition of a^r, for *any* $a, r \in E^1$ $(a > 0)$. We set

$$a^r = \exp(r \cdot \log a) \text{ or} \tag{7}$$

$$\log a^r = r \cdot \log a \quad (r \in E^1). \tag{8}$$

In case $r \in N$, (8) becomes Theorem 1(vi). Thus for natural r, our new definition of a^r is consistent with the previous one. We also obtain, for $a, b > 0$,

$$(ab)^r = a^r b^r; \quad a^{rs} = (a^r)^s; \quad a^{r+s} = a^r a^s; \quad (r, s \in E^1). \tag{9}$$

The proof is by taking logarithms. For example,

$$\begin{aligned}\log(ab)^r &= r \log ab = r(\log a + \log b) = r \cdot \log a + r \cdot \log b \\ &= \log a^r + \log b^r = \log(a^r b^r).\end{aligned}$$

Thus $(ab)^r = a^r b^r$. Similar arguments can be given for the rest of (9) and other laws stated in Chapter 2, §§11–12.

We can now define *the exponential to the base* a $(a > 0)$ and its inverse, $\log_a$, as before (see the example in Chapter 4, §5 and Example (b) in Chapter 4, §9). The differentiability of the former is now immediate from (7), and the rest follows as before.

II. Trigonometric Functions. These shall now be defined in a precise *analytic* manner (not based on geometry).

We start with an integral definition of what is usually called the *principal value of the arcsine function*,

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1 - t^2}}\, dt.$$

We shall denote it by $F(x)$ and set

$$f(x) = \frac{1}{\sqrt{1 - x^2}} \text{ on } I = (-1, 1).$$

$(F = f = 0$ on $E^1 - I.)$ Thus by definition, $F = \int f$ on I.

Note that $\int f$ exists and is *exact* on I since f is continuous on I. Thus

$$F'(x) = f(x) = \frac{1}{\sqrt{1-x^2}} > 0 \quad \text{for } x \in I,$$

and so F *is strictly increasing on* I. Also, $F(0) = \int_0^0 f = 0$.

We also define the number π by setting

$$\frac{\pi}{2} = 2 \arcsin \sqrt{\frac{1}{2}} = 2F(c) = 2\int_0^c f, \quad c = \sqrt{\frac{1}{2}}. \tag{10}$$

Then we obtain the following theorem.

Theorem 2. *F has the limits*

$$F(1^-) = \frac{\pi}{2} \text{ and } F(-1^+) = -\frac{\pi}{2}.$$

Thus F becomes relatively continuous on $\overline{I} = [-1, 1]$ if one sets

$$F(1) = \frac{\pi}{2} \text{ and } F(-1) = -\frac{\pi}{2},$$

i.e.,

$$\arcsin 1 = \frac{\pi}{2} \text{ and } \arcsin(-1) = -\frac{\pi}{2}. \tag{11}$$

Proof. We have

$$F(x) = \int_0^x f = \int_0^c f + \int_c^x f, \quad c = \sqrt{\frac{1}{2}}.$$

By substituting $s = \sqrt{1-t^2}$ in the last integral and setting, for brevity, $y = \sqrt{1-x^2}$, we obtain

$$\int_c^x f = \int_c^x \frac{1}{\sqrt{1-t^2}}\, dt = \int_y^c \frac{1}{\sqrt{1-s^2}}\, ds = F(c) - F(y). \quad \text{(Verify!)}$$

Now as $x \to 1^-$, we have $y = \sqrt{1-x^2} \to 0$, and hence $F(y) \to F(0) = 0$ (for F is continuous at 0). Thus

$$F(1^-) = \lim_{x \to 1^-} F(x) = \lim_{y \to 0} \left(\int_0^c f + \int_y^c f \right) = \int_0^c f + F(c) = 2\int_0^c f = \frac{\pi}{2}.$$

Similarly, one gets $F(-1^+) = -\pi/2$. □

The function F as redefined in Theorem 2 will be denoted by F_0. It is a primitive of f on the *closed* interval $\overline{I}$ (exact on I). Thus $F_0(x) = \int_0^x f$, $-1 \le x \le 1$, and we may now write

$$\frac{\pi}{2} = \int_0^1 f \text{ and } \pi = \int_{-1}^0 f + \int_0^1 f = \int_{-1}^1 f.$$

Note 2. In classical analysis, the last integrals are regarded as so-called *improper* integrals, i.e., *limits of integrals* rather than integrals proper. In our theory, this is unnecessary since F_0 is a *genuine* primitive of f on $\overline{I}$.

For each integer n (negatives included), we now define $F_n\colon E^1 \to E^1$ by

$$\begin{aligned} F_n(x) &= n\pi + (-1)^n F_0(x) \text{ for } x \in \overline{I} = [-1, 1], \\ F_n &= 0 \qquad\qquad\qquad\quad \text{on } -\overline{I}. \end{aligned} \tag{12}$$

F_n is called the *nth branch of the arcsine*. Figure 26 shows the graphs of F_0 and F_1 (that of F_1 is *dotted*). We now obtain the following theorem.

Theorem 3.

(i) *Each F_n is differentiable on $I = (-1, 1)$ and relatively continuous on $\overline{I} = [-1, 1]$.*

(ii) *F_n is increasing on $\overline{I}$ if n is even, and decreasing if n is odd.*

(iii) $F_n'(x) = \dfrac{(-1)^n}{\sqrt{1-x^2}}$ *on I.*

(iv) $F_n(-1) = F_{n-1}(-1) = n\pi - (-1)^n\dfrac{\pi}{2}$; $F_n(1) = F_{n-1}(1) = n\pi + (-1)^n\dfrac{\pi}{2}$.

The proof is obvious from (12) and the properties of F_0. Assertion (iv) ensures that the graphs of the F_n add up to *one* curve. By (ii), each F_n is one to one (strictly monotone) on $\overline{I}$. Thus it has a strictly monotone inverse on the interval $\overline{J_n} = F_n[[-1, 1]]$, i.e., on the F_n-image of $\overline{I}$. For simplicity, we consider only

$$\overline{J_0} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } J_1 = \left[\frac{\pi}{2}, \frac{3\pi}{2}\right],$$

as shown on the Y-axis in Figure 26. On these, we define for $x \in \overline{J_0}$

$$\sin x = F_0^{-1}(x) \tag{13}$$

and

$$\cos x = \sqrt{1-\sin^2 x}, \tag{13$'$}$$

and for $x \in \overline{J_1}$

$$\sin x = F_1^{-1}(x) \tag{14}$$

and

$$\cos x = -\sqrt{1-\sin^2 x}. \tag{14$'$}$$

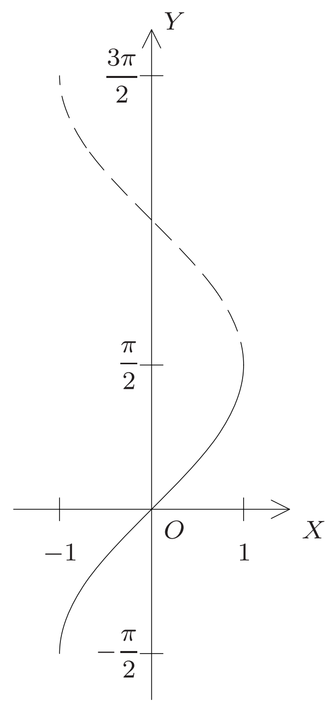

FIGURE 26

On the rest of E^1, we define $\sin x$ and $\cos x$ periodically by setting

$$\sin(x+2n\pi)=\sin x \text{ and } \cos(x+2n\pi)=\cos x, \quad n=0, \pm1, \pm2, \ldots. \tag{15}$$

Note that by Theorem 3(iv),

$$F_0^{-1}\left(\frac{\pi}{2}\right)=F_1^{-1}\left(\frac{\pi}{2}\right)=1.$$

Thus (13) and (14) both yield $\sin\pi/2=1$ for the common endpoint $\pi/2$ of $\overline{J_0}$ and $\overline{J_1}$, so the two formulas are consistent. We also have

$$\sin\left(-\frac{\pi}{2}\right)=\sin\left(\frac{3\pi}{2}\right)=-1,$$

in agreement with (15). Thus the sine and cosine functions (briefly, s and c) are well defined on E^1.

Theorem 4. *The sine and cosine functions* (s *and* c) *are differentiable*, *hence continuous*, *on all of* E^1, *with derivatives* $s'=c$ *and* $c'=-s$; *that is*,

$$(\sin x)'=\cos x \text{ and } (\cos x)'=-\sin x.$$

Proof. It suffices to consider the intervals $\overline{J_0}$ and $\overline{J_1}$, for, by (15), all properties of s and c repeat themselves, with period 2π, on the rest of E^1.

By (13),

$$s=F_0^{-1} \text{ on } \overline{J_0}=\left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

where F_0 is differentiable on $I=(-1, 1)$. Thus Theorem 3 of §2 shows that s is differentiable on $J_0=(-\pi/2, \pi/2)$ and that

$$s'(q)=\frac{1}{F_0'(p)} \text{ whenever } p\in I \text{ and } q=F_0(p);$$

i.e., $q\in J$ and $p=s(q)$. However, by Theorem 3(iii),

$$F_0'(p)=\frac{1}{\sqrt{1-p^2}}\,.$$

Hence

$$s'(q)=\sqrt{1-\sin^2 q}=\cos q=c(q), \quad q\in J.$$

This proves the theorem for *interior* points of $\overline{J_0}$ as far as s is concerned.

As

$$c=\sqrt{1-s^2}=(1-s^2)^{\frac{1}{2}} \text{ on } J_0 \text{ (by (13))},$$

we can use the chain rule (Theorem 3 in §1) to obtain

$$c'=\frac{1}{2}(1-s^2)^{-\frac{1}{2}}(-2s)s'=-s$$

on noting that $s' = c = (1 - s^2)^{\frac{1}{2}}$ on J_0. Similarly, using (14), one proves that $s' = c$ and $c' = -s$ on J_1 (interior of $\overline{J_1}$).

Next, let q be an *endpoint*, say, $q = \pi/2$. We take the *left* derivative

$$s'_-(q) = \lim_{x \to q^-} \frac{s(x) - s(q)}{x - q}, \quad x \in J_0.$$

By L'Hôpital's rule, we get

$$s'_-(q) = \lim_{x \to q^-} \frac{s'(x)}{1} = \lim_{x \to q^-} c(x)$$

since $s' = c$ on J_0. However, $s = F_0^{-1}$ is left continuous at q (why?); hence so is $c = \sqrt{1 - s^2}$. (Why?) Therefore,

$$s'_-(q) = \lim_{x \to q^-} c(x) = c(q), \quad \text{as required.}$$

Similarly, one shows that $s'_+(q) = c(q)$. Hence $s'(q) = c(q)$ and $c'(q) = -s(q)$, as before. $\square$

The other trigonometric functions reduce to s and c by their defining formulas

$$\tan x = \frac{\sin x}{\cos x}, \ \cot x = \frac{\cos x}{\sin x}, \ \sec x = \frac{1}{\cos x}, \ \text{and} \ \operatorname{cosec} x = \frac{1}{\sin x},$$

so we shall not dwell on them in detail. The various trigonometric laws easily follow from our present definitions; for hints, see the problems below.

Problems on Exponential and Trigonometric Functions

1. Verify formula (2).

2. Prove Note 1, as suggested (using Chapter 2, §§11–12).

3. Prove formulas (1) of Chapter 2, §§11–12 from our new definitions.

4. Complete the missing details in the proofs of Theorems 2–4.

5. Prove that

(i) $\sin 0 = \sin(n\pi) = 0$;

(ii) $\cos 0 = \cos(2n\pi) = 1$;

(iii) $\sin \frac{\pi}{2} = 1$;

(iv) $\sin\left(-\frac{\pi}{2}\right) = -1$;

(v) $\cos\left(\pm\frac{\pi}{2}\right) = 0$;

(vi) $|\sin x| \le 1$ and $|\cos x| \le 1$ for $x \in E^1$.

6. Prove that

(i) $\sin(-x) = -\sin x$ and

(ii) $\cos(-x) = \cos x$ for $x \in E^1$.

[Hint: For (i), let $h(x) = \sin x + \sin(-x)$. Show that $h' = 0$; hence h is constant, say, $h = q$ on E^1. Substitute $x = 0$ to find q. For (ii), use (13)–(15).]

7. Prove the following for $x, y \in E^1$:

(i) $\sin(x + y) = \sin x \cos y + \cos x \sin y$; hence $\sin\left(x + \frac{\pi}{2}\right) = \cos x$.

(ii) $\cos(x + y) = \cos x \cos y - \sin x \sin y$; hence $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$.

[Hint for (i): Fix x, y and let $p = x + y$. Define $h\colon E^1 \to E^1$ by

$$h(t) = \sin t \cos(p - t) + \cos t \sin(p - t), \quad t \in E^1.$$

Proceed as in Problem 6. Then let $t = x$.]

8. With $\overline{J_n}$ as in the text, show that the sine increases on $\overline{J_n}$ if n is even and decreases if n is odd. How about the cosine? Find the endpoints of $\overline{J_n}$.

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